# ADDING BOUNDS WHILE PRESERVING CONGRUENCES FOR LATTICES

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ABSTRACT. A lattice is *congruence-bounded* if its largest congruence is finitely generated. We study the following two statements for some varieties $\mathcal{V}$ of lattices.

(**Q1**) For every congruence-bounded lattice $K$ in $\mathcal{V}$ there is a bounded lattice $L \in \mathcal{V}$ such that $K$ and $L$ have isomorphic congruence lattices.

(**Q2**) Every congruence-bounded lattice in $\mathcal{V}$ has a bounded congruence-preserving extension in $\mathcal{V}$.

Given a finitely generated variety $\mathcal{V}$ of lattices, we prove that (**Q2**) holds only in the trivial case, that is if each congruence-bounded lattice in $\mathcal{V}$ is bounded. For example in $\mathcal{N}_5$, the variety generated by the five-element non-modular lattice, every congruence-bounded lattice is bounded.

Let $n \geq 3$. The statement (**Q2**) fails for the variety $\mathcal{M}_n$ generated by the lattice of length 2 with $n$ atoms, however (**Q1**) holds and the construction can be made functorial.

## 1. Introduction

The set $\operatorname{Con} L$ of all congruences of a lattice $L$ forms an algebraic lattice for the inclusion. We denote by $\Theta_L(x, y)$ the smallest congruence that identifies $x$ and $y$, for all $x, y$ in $L$. A congruence $\alpha$ of $L$ is *principal* if there are $x, y$ in $L$ such that $\alpha = \Theta(x, y)$. A *finitely generated* congruence is a finite join of principal congruences. The set $\operatorname{Con_c} L$ of all finitely generated congruences of $L$ forms a distributive $(\vee, 0)$-semilattice for the inclusion (cf. [2]).

A lattice is *congruence-bounded* if its largest congruence is compact (or equivalently finitely generated). Every bounded lattice is congruence-bounded but the converse does not hold in general. For example consider the lattice $\mathrm{M}_3$ on Figure 1, denote by $L$ the set of all sequences $(a_n)_{n<\omega}$ of elements of $\mathrm{M}_3$ such that either $\{n < \omega \mid a_n \neq 0\}$ is finite or $\{n < \omega \mid a_n \neq x\}$ is finite, endowed with the componentwise ordering. It is easy to check that $L$ is a lattice. Viewing $0$ and $x$ as constant sequences, $\Theta_L(0, x) = L \times L$ is the largest congruence of $L$. However $L$ has no largest element.

We know from [8] that there are distributive $(\vee, 0, 1)$-semilattices which are not isomorphic to the congruence lattice of any lattice. However many problems about congruence lattices of lattices are still open. The problem whether for each lattice $K$ there exists a lattice $L$ with $0$ such that $\operatorname{Con_c} K \cong \operatorname{Con_c} L$ is open (see [6, Problem 2] or the discussion before [7, Theorem 3.5]).

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The study of the following questions might be a good start to solve this problem. Let $S$ be a $(\vee, 0, 1)$-semilattice, we assume that there exists a lattice $K$ with $\mathrm{Con_c}\, K \cong S$. Is it possible to find a bounded lattice $L$ with $\mathrm{Con_c}\, L \cong S$? Can we choose $L$ to be a congruence-preserving extension of $K$?

We do not know the answers to those questions. However we study the following related questions for small varieties $\mathcal{V}$ of lattices.

(**Q1**) For all $K \in \mathcal{V}^{\mathrm{b}}$ there is $L \in \mathcal{V}^{0,1}$ such that $K$ and $L$ have isomorphic congruence lattices.

(**Q2**) Every $K \in \mathcal{V}^{\mathrm{b}}$ has a congruence-preserving extension in $\mathcal{V}^{0,1}$.

We prove in Corollary 3.8 that if each subdirectly irreducible lattice in $\mathcal{V}$ has no infinite chain, then (**Q2**) holds only in the trivial case, that is if $\mathcal{V}^{\mathrm{b}} = \mathcal{V}^{0,1}$.

Let $n \geq 3$, denote by $\mathcal{M}_n$ the variety of lattices generated by $M_n$ the lattice on Figure 2. We construct a functor $\Psi\colon \mathcal{M}_n^{\mathrm{b}} \to \mathcal{M}_n^{0,1}$, preserving colimits and such that $\mathrm{Con_c} \circ \Psi$ is naturally equivalent to $\mathrm{Con_c}$. In particular (**Q1**) holds for $\mathcal{M}_n$.

## 2. Basic Concepts

We denote by 0 (resp., 1) the least (resp. largest) element of a poset if it exists. We denote by $\mathbf{2} = \{0, 1\}$ the two-element lattice. We denote by $\mathbf{3}$ the three-element lattice. Given an algebra $A$, we denote by $\mathbf{0}_A$ (resp., $\mathbf{1}_A$) the identity congruence of $A$ (resp., the largest congruence of $A$).

Let $\mathcal{V}$ be a variety of lattices, we denote by $\mathcal{V}^0$ (resp., $\mathcal{V}^{0,1}$) the class of all lattices in $\mathcal{V}$ with 0 (resp., 0 and 1). We also consider $\mathcal{V}^0$ (resp., $\mathcal{V}^{0,1}$) as subcategories of $\mathcal{V}$ with morphisms preserving 0 (resp., 0 and 1).

Given a morphism $f\colon K \to L$ of lattices, we denote by $\mathrm{Con}\, f\colon \mathrm{Con}\, A \to \mathrm{Con}\, B$ the map that sends a congruence $\alpha$ of $K$ to the congruence of $L$ generated by $\{(f(x), f(y) \mid (x, y) \in \alpha\}$. We denote by $\mathrm{Con_c}\, f\colon \mathrm{Con_c}\, A \to \mathrm{Con_c}\, B$ the restriction of $\mathrm{Con}\, f$. Notice that $\mathrm{Con_c}$ is a functor from the category of lattices with morphisms of lattices to the category of $(\vee, 0)$-semilattice with $(\vee, 0)$-homomorphism.

The *kernel* $\ker f = \{(x, y) \in K^2 \mid f(x) = f(y)\}$ is a congruence of $K$, for any morphism of lattices $f\colon K \to L$. For $\beta \in \mathrm{Con}\, B$ we denote by $f^{-1}(\beta)$ the largest congruence $\alpha$ of $A$ such that $(\mathrm{Con_c}\, f)(\alpha) \subseteq \beta$, notice that $f^{-1}(\beta) = \{(x, y) \in A^2 \mid (f(x), f(y)) \in \beta\}$ is a congruence of $K$.

We denote by $M(L)$ the set of all meet-irreducible elements of a lattice $L$. Notice that $M(\mathrm{Con}\, A)$ is the set of all congruences $\alpha$ of a lattice $A$ such that $A/\alpha$ is subdirectly irreducible.

For a lattice $L$ and $a, b$ in $L$, we denote by $[a, b]_L$ the set of all $x$ in $L$ such that $a \leq x \leq b$. We say that $[a, b]_L$ is an *interval* of $L$. The *length* of a chain $C$ is $(\mathrm{card}\, C) - 1$. The *length* of a lattice $L$ is the maximal length of a chain contained in $L$.

If $K \subseteq L$ are lattices and $\alpha$ is a congruence of $L$ we identify $K/(\alpha \cap K^2)$ with the sublattice $K/\alpha = \{a/\alpha \mid a \in K\}$ of $L$. A *congruence-preserving extension* of a lattice $K$ is a lattice $L$ that contains $K$ such that any congruence of $K$ has a unique extension to $L$. Equivalently, $\mathrm{Con_c}\, f$ is an isomorphism, where $f\colon K \to L$ denotes the inclusion map. We also say that $K$ is a *congruence-preserving sublattice* of $L$.

For sets $X$ and $I$ we often denote $\vec{x} = (x_i \mid i \in I)$ an element of $X^I$. In particular, given $n < \omega$ we denote by $\vec{x} = (x_0, \ldots, x_{n-1})$ an $n$-tuple of $X$.

A nonempty poset $P$ is *directed* if for all $x, y \in P$ there exists $z \in P$ such that $z \geq x, y$.

## 3. Congruence-bounded lattices with bounded congruence-preserving extensions

The aim of this section is to study varieties of lattices in which each congruence-bounded lattice has a bounded congruence-preserving extension.

**Definition 3.1.** A lattice $L$ is *congruence-bounded* if $\mathbf{1}_L$ is a compact congruence.

*Notation* 3.2. Given a variety $\mathcal{V}$ of lattices, we denote by $\mathcal{V}^{\mathrm{b}}$ the category in which objects are congruence-bounded lattices of $\mathcal{V}$, and a morphism $f\colon A \to B$ in $\mathcal{V}^{\mathrm{b}}$ is a morphism of lattices such that $(\operatorname{Con_c} f)(\mathbf{1}_A) = \mathbf{1}_B$.

We refer to [4, Definition 1-3.1] or [1, Definitions 1.1 and 1.13] for the definition of finitely presented object.

**Lemma 3.3.** *Let $\mathcal{V}$ be a variety of algebras. The following statements hold*

1. *Let $P$ be a directed poset, let $\vec{A}$ be a $P$-indexed diagram in $\mathcal{V}^{\mathrm{b}}$. Let $(A, f_p \mid p \in P)$ be a colimit cocone of $\vec{A}$ in $\mathcal{V}$. Then $(A, f_p \mid p \in P)$ is a colimit cocone of $\vec{A}$ in $\mathcal{V}^{\mathrm{b}}$.*
2. *The subcategory $\mathcal{V}^{\mathrm{b}}$ of $\mathcal{V}$ is closed under small directed colimits.*
3. *If $\mathcal{V}$ is locally finite, then each finite algebra of $\mathcal{V}$ is a finitely presented object of $\mathcal{V}^{\mathrm{b}}$.*

*Proof.* Let $P$ be a directed poset, let $\vec{A} = (A_p, f_{p,q} \mid p \leq q \text{ in } P)$ be a diagram in $\mathcal{V}^{\mathrm{b}}$. Let $(A, f_p \mid p \in P)$ be a colimit cocone of $\vec{A}$ in $\mathcal{V}$.

Let $p \in P$, let $\alpha \in \operatorname{Con_c} A$. As $\operatorname{Con_c}$ preserves directed colimits, there is $q \geq p$ and $\beta \in \operatorname{Con} A_q$ such that $\alpha = (\operatorname{Con_c} f_q)(\beta)$, therefore

$$(\operatorname{Con_c} f_p)(\mathbf{1}_{A_p}) = (\operatorname{Con_c} f_q \circ f_{p,q})(\mathbf{1}_{A_p}) = (\operatorname{Con_c} f_q)(\mathbf{1}_{A_q}) \supseteq (\operatorname{Con_c} f_q)(\beta) = \alpha.$$

Thus $\operatorname{Con_c} A$ is bounded and $(\operatorname{Con_c} f_p)(\mathbf{1}_{A_p}) = \mathbf{1}_A$ for all $p \in P$. So $(A, f_p \mid p \in P)$ is a cocone of $\vec{A}$ in $\mathcal{V}^{\mathrm{b}}$.

Let $(B, g_p \mid p \in P)$ be a cocone of $\vec{A}$ in $\mathcal{V}^{\mathrm{b}}$, there is $g\colon A \to B$ a morphism in $\mathcal{V}$ such that $g \circ f_p = g_p$ for all $p \in P$. Let $p \in P$, as $(\operatorname{Con_c} f_p)(\mathbf{1}_{A_p}) = \mathbf{1}_A$ and $(\operatorname{Con_c} g_p)(\mathbf{1}_A) = \mathbf{1}_B$, it follows that $g$ is a morphism in $\mathcal{V}^{\mathrm{b}}$. Therefore $(A, f_p \mid p \in P)$ is a colimit cocone of $\vec{A}$ in $\mathcal{V}^{\mathrm{b}}$. So (1) holds.

Let $P$ be a directed poset. Let $\vec{A} = (A_p, f_{p,q} \mid p \leq q \text{ in } P)$ and $\vec{B} = (B_p, g_{p,q} \mid p \leq q \text{ in } P)$ be diagrams in $\mathcal{V}^{\mathrm{b}}$, together with colimit cocones in $\mathcal{V}$

$$(A, f_p \mid p \in P) = \varinjlim \vec{A}$$
$$(B, g_p \mid p \in P) = \varinjlim \vec{B}$$

Let $(h_p)_{p \in p}\colon \vec{A} \to \vec{B}$ be a natural transformation in $\mathcal{V}^{\mathrm{b}}$, denote by $h\colon A \to B$ the morphism of $\mathcal{V}$ such that $h \circ f_p = g_p \circ h_p$ for all $p \in P$. The following equalities hold

$$\begin{aligned}
(\operatorname{Con_c} h)(\mathbf{1}_A) &= (\operatorname{Con_c} h)((\operatorname{Con_c} f_p)(\mathbf{1}_{A_p})) && \text{as } f_p \text{ is a morphism in } \mathcal{V}^{\mathrm{b}}.\\
&= (\operatorname{Con_c} g_p)((\operatorname{Con_c} h_p)(\mathbf{1}_{A_p})) && \text{as } h \circ f_p = g_p \circ h_p.\\
&= (\operatorname{Con_c} g_p)(\mathbf{1}_{B_p}) && \text{as } h_p \text{ is a morphism in } \mathcal{V}^{\mathrm{b}}.\\
&= \mathbf{1}_B && \text{as } g_p \text{ is a morphism in } \mathcal{V}^{\mathrm{b}}.
\end{aligned}$$

Thus $h$ is a morphism in $\mathcal{V}^{\mathrm{b}}$. Therefore (2) holds.

Let $B$ be a finite algebra in $\mathcal{V}$, it follows that $\mathbf{1}_B$ is compact hence $B \in \mathcal{V}^{\mathrm{b}}$. Let $P$ be a directed poset. Let $\vec{A} = (A_p, f_{p,q} \mid p \leq q \text{ in } P)$ be a diagram in $\mathcal{V}^{\mathrm{b}}$, let $(A, f_p \mid p \in P)$ be a colimit cocone of $\vec{A}$ in $\mathcal{V}^{\mathrm{b}}$. Let $h\colon B \to A$ be a morphism in $\mathcal{V}^{\mathrm{b}}$.

As $B$ is finite, there exists $p \in P$ such that $h(B) \subseteq f_p(A_p)$. Let $X \subseteq A_p$ finite, such that $h(B) = f_p(X)$. As $\mathcal{V}$ is locally finite $C$ the subalgebra of $A_p$ generated by $X$ is finite. Moreover $h(B) = f_p(C)$. As $C$ is finite, there exists $q \geq p$, such that $f_q \restriction (f_{p,q}(C))$ is an embedding. Thus changing $p$ to $q$ and $C$ to $f_{p,q}(C)$ we can assume that $f_p \restriction C$ is an embedding.

As $C \cong f_p(C) = h(B)$, there is an isomorphism $k\colon h(B) \to C$ such that $k \circ f_p = \mathrm{id}_{f(B)}$, hence $f_p \circ k \circ h = h$, put $h' = k \circ h$. Notice that:

$$(\operatorname{Con_c} f_p)((\operatorname{Con_c} h')(\mathbf{1}_B)) = (\operatorname{Con_c} h)(\mathbf{1}_B) = \mathbf{1}_A = (\operatorname{Con_c} f_p)(\mathbf{1}_{A_p}),$$

therefore there is $q \geq p$ such that:

$$(\operatorname{Con_c} f_{p,q} \circ h')(\mathbf{1}_B) = (\operatorname{Con_c} f_{p,q})(\mathbf{1}_{A_p}) = \mathbf{1}_{A_q}.$$

Put $h'' = f_{p,q} \circ h'$, thus $(\operatorname{Con_c} h'')(\mathbf{1}B) = \mathbf{1}_{A_q}$, so $h''$ is a morphism in $\mathcal{V}^{\mathrm{b}}$. Moreover $f_q \circ h'' = f_q \circ f_{p,q} \circ h' = f_p \circ h' = f_p \circ k \circ h = h$. □

If all congruence-bounded lattices in a variety have congruence-preserving extensions with 0 then they have congruence-preserving extensions of bigger length.

**Lemma 3.4.** *Let $\mathcal{V}$ be a variety of lattices such that every countable lattices in $\mathcal{V}^{\mathrm{b}}$ has a congruence-preserving extension in $\mathcal{V}^0$. Let $A$ be a countable lattice in $\mathcal{V}^{\mathrm{b}}$, let $a < b < c$ in $A$ such that $\Theta_A(b, c) = \mathbf{1}_A$. There is a congruence-preserving extension $B \in \mathcal{V}$ of $A$ and $t \in B$ such that $t < a$.*

*Proof.* Denote by $A_0 = \{u, v\}$ a two element chain with $u < v$, put $A_1 = A$, put $A_2 = A$. Denote by $f_1\colon A_0 \to A_1$, $u \mapsto b$, $v \mapsto c$. Denote by $f_2\colon A_0 \to A_2$, $u \mapsto a$, $v \mapsto c$.

The following construction is a special case of condensate (cf. [3, 4]). However, in this case, the construction is simple; we give a self-contained proof.

Given an element $\vec{a} = (a_0, a_1^n, a_2^n \mid n \in \mathbb{N})$ of $A_0 \times A_1^{\mathbb{N}} \times A_2^{\mathbb{N}}$ we denote:

$$\operatorname{supp} \vec{a} = \{n \in \mathbb{N} \mid a_1^n \neq f_1(a_0) \text{ or } a_2^n \neq f_2(a_0)\}.$$

Given a finite subset $S$ of $\mathbb{N}$ we denote:

$$L_S = \{\vec{a} \in A_0 \times A_1^{\mathbb{N}} \times A_2^{\mathbb{N}} \mid \operatorname{supp} A \subseteq S\}.$$

Put $L = \bigcup(L_S \mid S \text{ is a finite subset of } \mathbb{N})$. Denote by $\pi_0\colon L \to A_0$, $\vec{a} \mapsto a_0$ and $\pi_k^n\colon L \to A_k$, $\vec{a} \mapsto a_k^n$ for each $k \in \{1, 2\}$ and each $n \in \mathbb{N}$.

**Claim 1.** *The lattice $L$ is countable and belongs to $\mathcal{V}^{\mathrm{b}}$.*

*Proof of Claim.* Let $S$ be a finite subset of $\mathbb{N}$. Notice that the restriction map $L_S \to A_0 \times A_1^S \times A_2^S$, $\vec{a} \mapsto (a_0, a_1^n, a_2^n \mid n \in S)$ is an isomorphism. Hence $L$ is countable (as a countable union of countable lattices). Moreover the following equality holds

$$\ker(\pi_0 \restriction L_S) \cap \bigcap_{n \in S} \ker(\pi_1^n \restriction L_S) \cap \bigcap_{n \in S} \ker(\pi_2^n \restriction L_S) = \mathbf{0}_{L_S}. \tag{3.1}$$

Let $\vec{u} = (u, f_1(u), f_2(u) \mid n \in \mathbb{N})$, $\vec{v} = (v, f_1(v), f_2(v) \mid n \in \mathbb{N})$. Let $S$ be a finite subset of $\mathbb{N}$. Notice that $\vec{u}, \vec{v}$ belong to $L_S$, moreover the following equalities hold

$$(\operatorname{Con} \pi_0 \restriction L_S)(\Theta_{L_S}(\vec{u}, \vec{v})) = \Theta_{A_0}(u, v) = \mathbf{1}_{A_0}.$$

It implies that

$$\ker(\pi_0 \restriction L_S) \vee \Theta_{L_S}(\vec{u},\vec{v}) = \mathbf{1}_{L_S}. \tag{3.2}$$

Similarly, given $k \in \{1,2\}$ and $n \in \mathbb{N}$ the following equalities hold

$$(\operatorname{Con} \pi_k^n \restriction L_S)(\Theta_{L_S}(\vec{u},\vec{v})) = \Theta_{A_k^n}(f_k(u), f_k(v)) = \mathbf{1}_{A_k^v}.$$

Thus we obtain

$$\ker(\pi_k^n \restriction L_S) \vee \Theta_{L_S}(\vec{u},\vec{v}) = \mathbf{1}_{L_S}. \tag{3.3}$$

Put $\theta_0 = \ker(\pi_0 \restriction L_S)$, put $\theta_k^n = \ker(\pi_k^n \restriction L_S)$ for all $k \in \{1,2\}$ and all $n \in \mathbb{N}$, put $\alpha = \Theta_{L_S}(\vec{u},\vec{v})$. The following equalities hold

$$\begin{aligned}
\Theta_{L_S}(\vec{u},\vec{v}) &= \alpha \vee \left(\theta_0 \cap \bigcap_{n\in S} \theta_1^n \cap \bigcap_{n\in S} \theta_2^n\right), && \text{by (3.1).}\\
&= (\alpha \vee \theta_0) \cap \bigcap_{n\in S}(\alpha \vee \theta_1^n) \cap \bigcap_{n\in S}(\alpha\vee\theta_2^n), && \text{by distributivity.}\\
&= \mathbf{1}_{L_S}, && \text{by (3.2) and (3.3).}
\end{aligned}$$

As $L = \bigcup(L_S \mid S \text{ is a finite subset of } \mathbb{N})$, it follows that $\mathbf{1}_L = \Theta_L(\vec{u},\vec{v})$ is a compact congruence of $L$, hence $L$ belongs to $\mathcal{V}^{\mathrm{b}}$. □ Claim 1.

**Claim 2.** *Let* $\theta \in \operatorname{Con_c} L$. *There is* $n \in \mathbb{N}$ *such that* $(\operatorname{Con_c} f_k \circ \pi_0)(\theta) = (\operatorname{Con_c} \pi_k^n)(\theta)$ *for* $k \in \{1,2\}$.

*Proof of Claim.* We first assume that $\theta$ is principal, there is $\vec{a},\vec{b}$ in $L$ such that $\theta = \Theta_L(\vec{a},\vec{b})$. We prove that the equality holds for all $n$ except finitely many. Let $n \in \mathbb{N} - (\operatorname{supp}(\vec{a}) \cup \operatorname{supp}(\vec{b}))$, let $k \in \{1,2\}$. The following equalities hold

$$\begin{aligned}
(\operatorname{Con_c} f_k \circ \pi_0)(\Theta_L(\vec{a},\vec{b})) &= \Theta_{A_k}(f_k(\pi_k(\vec{a})), f_k(\pi_k(\vec{b})))\\
&= \Theta_{A_k}(f_k(a_0), f_k(b_0))\\
&= \Theta_{A_k}(a_n, b_n), && \text{as } n \notin \operatorname{supp}(\vec{a}) \cup \operatorname{supp}(\vec{b}).\\
&= \Theta_{A_k}(\pi_k^n(\vec{a}), \pi_k^n(\vec{b}))\\
&= (\operatorname{Con_c} \pi_k^n)(\Theta_L(\vec{a},\vec{b}))
\end{aligned}$$

As a compact congruence is a finite join of principal congruences the conclusion follows. □ Claim 2.

Let $K \in \mathcal{V}^0$ be a congruence-preserving extension of $L$. We identify $\operatorname{Con_c} K$ and $\operatorname{Con_c} L$. Denote by $p_0 \colon K \twoheadrightarrow K/\ker \pi_0$ and by $p_k^n \colon K \twoheadrightarrow K/\ker \pi_k^n$ the canonical projections. Denote by $i_k^n \colon A_k^n \hookrightarrow K/\ker \pi_k^n$ the morphism induced by the inclusion map $L \hookrightarrow K$, that is the morphism such that $i_k^n \circ \pi_k^n = p_k^n$. Notice that $\operatorname{Con_c} i_k^n$ is an isomorphism, for all $k \in \{1,2\}$ and all $n \in \mathbb{N}$.

Put $\vec{u} = (u, f_1(u), f_2(u) \mid n \in \mathbb{N})$. Put $\theta = \Theta_K(0,\vec{u})$. From Claim 1 there is $n \in \mathbb{N}$ such that $(\operatorname{Con_c} f_k \circ \pi_0)(\theta) = (\operatorname{Con_c} \pi_k^n)(\theta)$, for all $k \in \{1,2\}$.

If $\theta \subseteq \ker \pi_0$ then $(\operatorname{Con_c} \pi_1^n)(\theta) = (\operatorname{Con_c} f_1 \circ \pi_0)(\theta) = \mathbf{0}_{A_1}$. Therefore the following equalities hold

$$\begin{aligned}
i_1^n(f_1(u)) &= i_1^n(\pi_1^n(\vec{u})), && \text{by definition of } \vec{u}.\\
&= p_1^n(\vec{u}), && \text{as } i_1^n \text{ is induced by the inclusion map.}\\
&= p_1^n(0), && \text{as } (0,\vec{u}) \in \ker p_1^n = \ker \pi_1^n.
\end{aligned}$$

However $a < b = f_1(u)$, so $i_1^n(a) < i_1^n(f_1(u)) = p_1^n(0) \leq i_1^n(a)$; a contradiction. It follows that $\theta \nsubseteq \ker p_0$.

Notice that $(\mathrm{Con_c}\, f_2 \circ \pi_0)(\theta) = (\mathrm{Con_c}\, \pi_2^n)(\theta)$. As $\theta \nsubseteq \ker p_0$ and $\mathrm{Con_c}\, f_2$ separates 0, it follows that $(\mathrm{Con_c}\, \pi_2^n)(\theta) \neq \mathbf{0}_{A_2}$, hence:

$$p_2^n(0) < p_2^n(\vec{u}) = i_2^n(\pi_2^n(\vec{u})) = i_2^n(f_2(u)) = i_2^n(a).$$

As $i_2^n$ is an isomorphism we can identify $L/\ker \pi_2^n$ with a congruence-preserving extension of $A = A_2$, put $t = p_2^n(0)$, we have $t < a$. □

*Remark* 3.5. In the context of Lemma 3.4, changing $B$ to one of its sublattices, we can assume that $B$ is countable (cf. [3, Lemma 3.6]).

**Theorem 3.6.** *Let $\mathcal{V}$ be a variety of lattices. If each subdirectly irreducible lattice in $\mathcal{V}$ has no infinite decreasing sequence then the following statements are equivalent.*

1. *Each countable congruence-bounded lattice in $\mathcal{V}$ has a congruence-preserving extension in $\mathcal{V}^0$.*
2. *Let $L \in \mathcal{V}$ be a subdirectly irreducible lattice, let $x < y$ in $L$. If $\Theta_L(x, y) = \mathbf{1}_L$ then $x = 0$.*
3. *Let $L \in \mathcal{V}$, let $x < y$ in $L$. If $\Theta_L(x, y) = \mathbf{1}_L$ then $x = 0$.*
4. *The category $\mathcal{V}^{\mathrm{b}}$ is a subcategory of $\mathcal{V}^0$.*

*Proof.* The implication (4) $\Longrightarrow$ (1) is immediate.

Assume that (1) holds and (2) fails. There is a subdirectly irreducible lattice $L \in \mathcal{V}$ and elements $a < b < c$ in $L$ such that $\Theta_L(b, c) = \mathbf{1}_L$. Changing $L$ to one of its sublattice we can assume that $L$ is countable. Put $L_0 = L$ and $a_0 = a$.

Let $n > 0$. Assume that we have constructed a sequence $(L_i)_{i<n}$ of countable lattices and a sequence $(a_i)_{i<n}$ such that $a_i \in L_i$, the lattice $L_{i+1}$ is a congruence-preserving extension of $L_i$, and $a_{i+1} < a_i$, for all $i < n - 1$.

Notice that $a_{n-1} \leq a_0 = a < b < c$. By Lemma 3.4 there is a congruence-preserving extension $L_n$ of $L_{n-1}$ and $a_n \in L_n$ such that $a_n < a_{n-1}$. Moreover by Remark 3.5 we can assume that $L_n$ is countable. Hence we construct by induction a sequence $(L_i)_{i<\omega}$ of countable lattices and a sequence $(a_i)_{i<\omega}$ such that $a_i \in L_i$, the lattice $L_{i+1}$ is a congruence-preserving extension of $L_i$, and $a_{i+1} < a_i$, for all $i < \omega$.

Put $K = \bigcup_{n<\omega} L_i$, as $\mathrm{Con_c}$ preserves directed colimits it follows that $\mathrm{Con_c}\, K \cong \mathrm{Con_c}\, L_0$ therefore $K$ is subdirectly irreducible, moreover the $a_i$s form an infinite decreasing sequence; a contradiction.

Assume that (2) is satisfied, let $L \in \mathcal{V}^{\mathrm{b}}$, let $x < y$ in $L$ such that $\Theta_L(x, y) = \mathbf{1}_L$. Let $\alpha \in M(\mathrm{Con}\, L)$, then $L/\alpha$ is subdirectly irreducible, moreover $\Theta_{L/\alpha}(x/\alpha, y/\alpha) = \mathbf{1}_{L/\alpha}$, so it follows from (2) that $x/\alpha = 0$.

Thus $x/\alpha = 0$ for all $\alpha \in M(\mathrm{Con}\, L)$, however $\bigcap M(\mathrm{Con}\, L) = \mathbf{0}_L$, hence $x = 0$. Therefore (2) $\Longrightarrow$ (3). Assume that (3) holds, let $L$ in $\mathcal{V}^{\mathrm{b}}$, there are $x, y$ in $L$ such that $\Theta_L(x, y) = \mathbf{1}_L$, it follows from (3) that $L$ has 0. Let $f\colon K \to L$ be a morphism in $\mathcal{V}^{\mathrm{b}}$, there are $x, y$ in $K$ such that $\Theta_K(x, y) = \mathbf{1}_K$. The following equalities hold:

$$\Theta_L(f(x), f(y)) = (\mathrm{Con_c}\, f)(\Theta_K(x, y)) = (\mathrm{Con_c}\, f)(\mathbf{1}_K) = \mathbf{1}_L.$$

Therefore, from (3) we obtain $f(0) = f(x) = 0$, therefore $f$ is a morphism in $\mathcal{V}^0$. □

*Remark* 3.7. There are lattices without any congruence-preserving extension with 0 (in any variety of lattices). For example consider an infinite chain $A = \{x_0 > x_1 >$

$x_2 > \dots\}$. Notice that $\mathrm{Con_c}\, A$ has no largest element. Let $B$ be a congruence-preserving extension of $A$ with 0, we identify $\mathrm{Con}\, A$ and $\mathrm{Con}\, B$. The containment $\Theta_B(0,x_0) \supseteq \Theta_B(x_k,x_0)$ holds for all $k<\omega$. Therefore $\mathbf{1}_B \supseteq \Theta_B(0,x_0) \supseteq \mathbf{1}_A = \mathbf{1}_B$ hence $\mathbf{1}_B = \Theta_B(0,x_0)$ is compact; a contradiction.

The following corollary is an immediate consequence of Theorem 3.6 and its dual.

**Corollary 3.8.** *Let $\mathcal{V}$ be a variety of lattices. If each subdirectly irreducible lattices in $\mathcal{V}$ has no infinite chain then the following statements are equivalent.*

1. *Each countable lattice in $\mathcal{V}^{\mathrm{b}}$ has a congruence-preserving extension in $\mathcal{V}^{0,1}$.*
2. *Let $L \in \mathcal{V}$ be a subdirectly irreducible lattice, let $x<y$ in $L$. If $\Theta_L(x,y) = \mathbf{1}_L$ then $x=0$ and $y=1$.*
3. *The equality $\mathcal{V}^{\mathrm{b}} = \mathcal{V}^{0,1}$ holds (the two categories have the same objects and the same morphisms).*

**Example 3.9.** Denote by $\mathcal{M}_3$ the variety of lattices generated by $M_3$ (see Figure 1). Notice that $\Theta_{M_3}(x,1) = \mathbf{1}_{M_3}$, so $M_3$ fails the condition Theorem 3.6(2) hence there is a congruence-bounded lattice $L$ in $\mathcal{M}_3$ with no congruence-preserving extension with 0 in $\mathcal{M}_3$.

Denote by $\mathcal{N}_5$ the variety of lattices generated by $N_5$ (see Figure 1). The subdirectly irreducible lattices of $\mathcal{N}_5$ are, up to isomorphisms, $N_5$ and $\mathbf{2}$, they satisfy both the condition Corollary 3.8(2). Therefore each congruence-bounded lattice in $\mathcal{N}_5$ has 0 and 1.

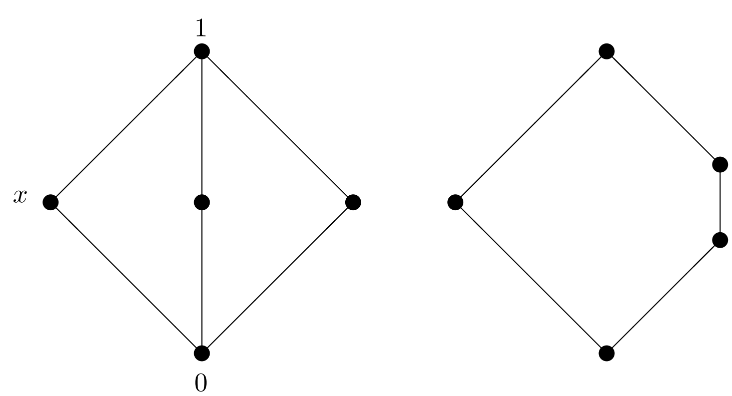


FIGURE 1. The lattices $M_3$ and $N_5$.

## 4. A functor

The goal of this section is to construct a functor $\Psi\colon \mathcal{M}_n^{\mathrm{b}} \to \mathcal{M}_n^{0,1}$, preserving colimits and such that $\mathrm{Con_c} \circ \Psi$ is naturally equivalent to $\mathrm{Con_c}$.

The following Lemma expresses that an interval of a quotient of a lattice is a quotient of an interval (we identify a quotient of a sublattice with a sublattice of a quotient).

**Lemma 4.1.** *Let $a \le b$ in a lattice $L$, let $\theta \in \mathrm{Con}\, L$, then $[a,b]_L/\theta = [a/\theta, b/\theta]_{L/\theta}$.*

*Proof.* Let $x \in [a,b]_L/\theta$, there is $y \in [a,b]_L$ such that $y/\theta = x$, thus $a/\theta \leq y/\theta = x \leq b/\theta$, therefore $x \in [a/\theta, b/\theta]_{L/\theta}$.

Let $x \in [a/\theta, b/\theta]_{L/\theta}$, there is $y \in L$ such that $y/\theta = x$. Put $y' = (y \vee a) \wedge b$, then $a \leq y' \leq b$, and $y'/\theta = (y/\theta \vee a/\theta) \wedge b/\theta = x$, therefore $x \in [a,b]_L/\theta$. □

*Remark* 4.2. Let $K$ be a simple lattice in $\mathcal{M}_\omega$, let $x < y$ in $K$. The lattice $[x,y]_K$ is simple.

**Lemma 4.3.** *Let $K$ be a simple lattice in $\mathcal{M}_\omega$, let $x < y$ in $K$, let $u \leq v$ and $u' \leq v'$ in $[x,y]_K$. If $\Theta_K(u,v) = \Theta_K(u',v')$ then $\Theta_{[x,y]_K}(u,v) = \Theta_{[x,y]_K}(u',v')$*

*Proof.* As $\Theta_K(u,v) = \Theta_K(u',v')$ it follows that $u = v$ if and only if $u' = v'$.

If $u = v$ then $\Theta_{[x,y]_K}(u,v) = \mathbf{0}_{[x,y]_K} = \Theta_{[x,y]_K}(u',v')$. If $u \neq v$, as $[x,y]_K$ is simple (cf. Remark 4.2), it follows that $\Theta_{[x,y]_K}(u,v) = \mathbf{1}_{[x,y]_K} = \Theta_{[x,y]_K}(u',v')$. □

*Remark* 4.4. Let $L$ be a finite modular lattice then $\operatorname{Con} L$ is a Boolean semilattice, moreover the atoms of $\operatorname{Con} L$ are the congruences of the form $\Theta_L(u,v)$ for $u \prec v$ in $L$.

**Lemma 4.5.** *Let $L$ be a finite lattice in $\mathcal{M}_\omega$, let $x < y$ in $L$. Denote by $f\colon [x,y]_L \hookrightarrow L$ the inclusion map. The restriction $\operatorname{Con} f\colon \operatorname{Con}[x,y]_L \to \operatorname{Con} L \downarrow \Theta_L(x,y)$ is an isomorphism.*

*Proof.* Put $A = [x,y]_L$, as $A$ and $L$ are both finite modular lattices, it follows that $\operatorname{Con} A$ and $\operatorname{Con} L$ are finite Boolean semilattice.

Let $\theta$ be an atom of $\operatorname{Con} A$. By Remark 4.4 there are $u \prec v$ in $A$ such that $\theta = \Theta_A(u,v)$, however $u \prec v$ in $L$, it follows from Remark 4.4 that $\Theta_L(u,v)$ is an atom of $\operatorname{Con} L$. Thus $(\operatorname{Con} f)(\theta) = \Theta_L(u,v)$ is an atom of $\operatorname{Con} L$.

The following equalities hold

$$\begin{aligned}
\bigvee (\operatorname{Con} f)(\operatorname{At}(\operatorname{Con} A)) &= (\operatorname{Con} f)\Big(\bigvee \operatorname{At}(\operatorname{Con} A)\Big)\\
&= (\operatorname{Con} f)(\mathbf{1}_A)\\
&= (\operatorname{Con} f)(\Theta_A(x,y))\\
&= \Theta_L(x,y).
\end{aligned}$$

As $\operatorname{Con} L$ is a Boolean semilattice and $(\operatorname{Con} f)(\operatorname{At}(\operatorname{Con} A)) \subseteq \operatorname{At}(\operatorname{Con} L)$, it follows that $(\operatorname{Con} f)(\operatorname{At}(\operatorname{Con} A)) = \operatorname{At}(\operatorname{Con} L \downarrow \Theta_L(x,y))$. Therefore the restriction of $\operatorname{Con} f$ is surjective.

To prove that $\operatorname{Con} f$ is one-to-one we just have to prove that $(\operatorname{Con} f) \restriction \operatorname{At}(\operatorname{Con} L)$ is one-to-one. Let $\alpha, \beta \in \operatorname{At}(\operatorname{Con} L)$. Assume that $(\operatorname{Con} f)(\alpha) = (\operatorname{Con} f)(\beta)$. Let $u \prec v$ in $A$ such that $\alpha = \Theta_A(u,v)$, let $u' \prec v'$ in $A$ such that $\beta = \Theta_A(u',v')$. Notice that $\Theta_L(u,v) = (\operatorname{Con} f)(\alpha) = (\operatorname{Con} f)(\beta) = \Theta_L(u',v')$.

Let $\theta \in M(\operatorname{Con} L)$. If $x/\theta = y/\theta$, then $u/\theta = v/\theta = u'/\theta = v'/\theta$, hence $\Theta_{A/\theta}(u/\theta, v/\theta) = \Theta_{A/\theta}(u'/\theta, v'/\theta)$.

Now we assume that $x/\theta < y/\theta$. By Lemma 4.1, $A/\theta$ is an interval in $L/\theta$. Notice that $\Theta_{L/\theta}(u/\theta, v/\theta) = \Theta_{L/\theta}(u'/\theta, v'/\theta)$, moreover $L/\theta$ is simple (see Remark 4.4), therefore Lemma 4.3 implies that $\Theta_{A/\theta}(u/\theta, v/\theta) = \Theta_{A/\theta}(u'/\theta, v'/\theta)$. Thus the following equality holds

$$\Theta_A(u,v) \vee (\theta \cap A^2) = \Theta_A(u',v') \vee (\theta \cap A^2), \quad \text{for each } \theta \in M(\operatorname{Con} L).$$

However $M(\operatorname{Con} L)$ is finite, $\operatorname{Con} A$ is distributive and $\bigwedge M(\operatorname{Con} L) = 0_L$, hence $\alpha = \Theta_A(u,v) = \Theta_A(u',v') = \beta$. Therefore $\operatorname{Con_c} f$ is one-to-one. □

The following corollary is an immediate consequence of Lemma 4.5.

**Corollary 4.6.** *Let $L$ be a finite lattice in $\mathcal{M}_\omega$, let $a \leq b \leq c \leq d$ in $L$ such that $\Theta_L(b,c) = \Theta_L(a,d)$. Then $\Theta_{[a,d]_L}(b,c) = \mathbf{1}_{[a,d]_L}$.*

*Remark* 4.7. Denote by $\mathcal{M}_{3,3}$ the variety of lattices generated by $M_{3,3}$, see Figure 2. Corollary 4.6 cannot be generalized for $\mathcal{M}_{3,3}$. We have $\Theta_{M_{3,3}}(b,c) = \mathbf{1}_{M_{3,3}}$ and $\Theta_{[a,d]}(b,c) \neq \mathbf{1}_{[a,d]}$.

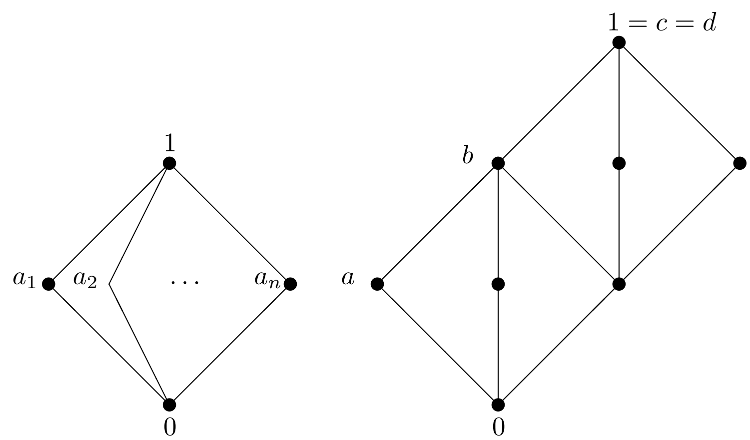


FIGURE 2. The lattices $M_n$ and $M_{3,3}$.

The following result appears in [5, Theorem 10.4]. It gives a description of finitely generated congruences of general algebra.

**Lemma 4.8.** *Let $B$ be an algebra, let $m$ be a positive integer, let $x, y \in B$, and let $\vec{x}, \vec{y}$ be $m$-tuples of $B$. Then $\Theta_B(x,y) \leq \bigvee_{i<m} \Theta_B(x_i, y_i)$ if and only if there are a positive integer $n$, a list $\vec{z}$ of parameters from $B$, and terms $t_0, \dots, t_n$ such that*

$$\begin{aligned} x &= t_0(\vec{x}, \vec{y}, \vec{z}), \\ y &= t_n(\vec{x}, \vec{y}, \vec{z}), \\ t_j(\vec{y}, \vec{x}, \vec{z}) &= t_{j+1}(\vec{x}, \vec{y}, \vec{z}) \quad (\textit{for all } j < n). \end{aligned}$$

It follows from Lemma 4.8 that if two finitely generated congruences are comparable in a locally finite algebra, then there is a "reason" in a finite subalgebra.

**Corollary 4.9.** *Let $B$ be a locally finite algebra, let $m$ be a positive integer, let $X$ be a finite subset of $B$, let $x, y \in X$, and let $\vec{x}, \vec{y}$ be $m$-tuples of $X$, if $\Theta_B(x,y) \leq \bigvee_{i<m} \Theta_B(x_i, y_i)$, then there is a finite subalgebra $C$ of $B$ such that $X \subseteq C$ and $\Theta_C(x,y) \leq \bigvee_{i<m} \Theta_C(x_i, y_i)$.*

**Lemma 4.10.** *Let $L \in \mathcal{M}_\omega^{\mathrm{b}}$. Let $x < y$ in $L$ such that $\Theta_L(x,y) = \mathbf{1}_L$. Denote by $P$ the set of all finite sublattice $A$ of $L$ such that $x, y \in A$ and $\Theta_A(x,y) = \mathbf{1}_A$. The following statement are satisfied*

1. *Let $X$ be a finite subset of $L$, there is $A \in P$ such that $X \subseteq A$.*
2. *The poset $(P, \subseteq)$ is directed.*

*Proof.* Let $X$ be a finite subset of $L$, we can assume that $x, y \in X$. Put $a = \bigwedge X$, put $b = \bigvee X$, hence $a \leq x < y \leq b$. Notice that $\Theta_L(a,b) = \Theta_L(x,y) = \mathbf{1}_L$. It follows from Corollary 4.9 that there exists a finite sublattice $A$ of $L$ such that $X \subseteq A$ and $\Theta_A(a,b) \subseteq \Theta_A(x,y)$. However $a \leq x < y \leq b$ so $\Theta_A(a,b) = \Theta_A(x,y)$. It follows from Corollary 4.6 that $\Theta_{[a,b]_A}(x,y) = \mathbf{1}_{[a,b]_A}$.

Put $B = [a,b]_A$. Notice that $X \subseteq B$ and $\Theta_B(x,y) = \mathbf{1}_B$, therefore $B$ belongs to $P$. Hence (1) holds.

The statement (2) follows from (1). $\square$

From Lemma 4.10 and Lemma 3.3 we obtain the following corollary.

**Corollary 4.11.** *Let $n \leq \omega$. Every lattice in $\mathcal{M}_n^{\mathrm{b}}$ is a directed colimit of finite lattices in $\mathcal{M}_n^{\mathrm{b}}$. The finitely presented objects in $\mathcal{M}_n^{\mathrm{b}}$ are the finite lattices in $\mathcal{M}_n^{\mathrm{b}}$.*

Each lattice in $\mathcal{M}_\omega^{\mathrm{b}}$ has a bounded congruence-preserving sublattice. It is a generalization of Lemma 4.5 in the infinite case.

**Corollary 4.12.** *Let $L \in \mathcal{M}_\omega^{\mathrm{b}}$. Let $x < y$ in $L$ such that $\mathbf{1}_L = \Theta_L(x,y)$. Denote by $f\colon [x,y]_L \hookrightarrow L$ the inclusion map, then $\operatorname{Con_c} f$ is an isomorphism.*

*Proof.* Denote by $P$ the set of all finite sublattice $A$ of $L$, such that $x, y \in A$ and $\Theta_A(x,y) = \mathbf{1}_A$. Lemma 4.10 implies that $(P, \subseteq)$ is a directed poset, moreover $L = \bigcup_{A \in P} A$. It follows that $[x,y]_L = \bigcup_{A \in P} [x,y]_A$.

Denote by $f_A\colon [x,y]_A \to A$ the inclusion map. As $\Theta_A(x,y) = \mathbf{1}_A$, it follows from Lemma 4.5 that $\operatorname{Con} f_A\colon \operatorname{Con}[x,y]_A \to \operatorname{Con} A$ is an isomorphism, for all $A \in P$. Moreover $f = \bigcup_{A \in P} f_A$. Therefore $\operatorname{Con} f = \varinjlim \operatorname{Con} f_A$, but $\operatorname{Con} f_A$ is an isomorphism for all $A \in P$, therefore $\operatorname{Con} f$ is an isomorphism. $\square$

Corollary 4.12 extends to diagrams indexed by poset with 0.

**Corollary 4.13.** *Let $P$ be a poset with 0. Let $\vec{A} = (A_p, f_{p,q} \mid p \leq q \text{ in } P)$ be a diagram in $\mathcal{M}_\omega^{\mathrm{b}}$. Let $x < y$ in $A_0$ such that $\mathbf{1}_{A_0} = \Theta_{A_0}(x,y)$. Put $B_p = [f_{0,p}(x), f_{0,p}(y)]_{A_p}$, denote by $t_p\colon B_p \hookrightarrow A_p$ the inclusion map, denote by $g_{p,q}\colon B_p \to B_q$ the restriction of $f_{p,q}$, for all $p \leq q$ in $P$. Put $\vec{B} = (B_p, g_{p,q} \mid p \leq q \text{ in } P)$, it is a diagram in $\mathcal{M}_\omega^{0,1}$. The family $(t_p)_{p \in P}$ is a natural transformation from $\vec{B}$ to $\vec{A}$. Moreover $(\operatorname{Con_c} t_p)_{p \in P}$ is a natural equivalence.*

*Remark* 4.14. The Corollary 4.13 cannot be extended to diagrams indexed by arbitrary poset. We consider the sublattices $A_0 = \{0, x\}$ and $A_1 = \{x, 1\}$ of $M_3$ (see Figure 1). The three lattices $A_0, A_1, M_3$ form a diagram $\vec{A}$ of $\mathcal{M}_3^{\mathrm{b}}$ under inclusion. The diagram $\vec{A}$ is not a congruence-preserving extension of any diagram in $\mathcal{M}_3^{0,1}$.

The following Lemma is proved in [3, Lemma 8.1].

**Lemma 4.15.** *Let $A$ be a finite algebra with $\operatorname{Con} A$ distributive, let $\alpha \in \operatorname{Con} A$, and put $Q = \{\theta \in \mathrm{M}(\operatorname{Con} A) \mid \alpha \not\leq \theta\}$. If all $A/\theta$, for $\theta \in Q$, are simple, then the canonical map $\operatorname{Con} A \to \operatorname{Con}(A/\alpha) \times \prod_{\theta \in Q} \operatorname{Con}(A/\theta)$ is an isomorphism.*

*Notation* 4.16. Let $3 \leq n \leq \omega$, we denote by $\mathcal{M}_n^{\mathrm{b}\dagger}$ the full subcategory of $\mathcal{M}_n^{\mathrm{b}}$ in which objects are the finite lattices in $\mathcal{M}_n^{\mathrm{b}}$.

*Remark* 4.17. Let $f\colon A \to B$ a morphism of distributive lattices. If $(\operatorname{Con_c} f)(\mathbf{1}_A) = \mathbf{1}_B$, then $f$ is a $0,1$-homomorphism.

**Lemma 4.18.** *There is a functor* $\Psi\colon \mathcal{M}_n^{\mathrm{b}\dagger} \to \mathcal{M}_n^{0,1}$ *such that* $\mathrm{Con_c} \circ \Psi$ *is naturally equivalent to* $\mathrm{Con_c}$.

*Proof.* Let $A \in \mathcal{M}_n$ be a finite lattice, we denote by $\alpha_A$ the smallest congruence of $A$ such that $A/\alpha_A$ is distributive. Denote $Q_A = \{\theta \in \mathrm{M}(\mathrm{Con}\, A) \mid \alpha_A \nsubseteq \theta\}$. Notice that if $\beta \in Q_A$, then $A/\beta$ is simple and not distributive, thus $A/\beta \in \{M_k \mid 3 \leq k < \omega\}$. Denote $R_A = \{\alpha_A\} \cup Q_A$. Denote $S_A = \{\theta \in \mathrm{Con}\, A \mid \theta \supseteq \alpha_A \text{ or } \theta \in R_A\}$.

Put $\Psi(A) = \prod_{\beta \in R_A} A/\beta$. Denote by $t_A\colon A \to \Psi(A)$, $x \mapsto (x/\beta)_{\beta\in R_A}$. Put $\xi_A = \mathrm{Con}\, t_A$, by Lemma 4.15 the map $\xi_A$ is an isomorphism.

Given $\theta \in S_A$, we denote

$$\begin{aligned} p_\theta^A\colon \Psi(A) &\to A/\theta \\ (u_\beta/\beta)_{\beta\in R_A} &\mapsto \begin{cases} u_\theta/\theta & \text{if } \theta \in R_A. \\ u_{\alpha_A}/\theta & \text{if } \theta \supseteq \alpha_A. \end{cases} \end{aligned}$$

Given $\theta \subseteq \gamma$ in $S_A$, we denote by $p_{\theta,\gamma}^A\colon A/\theta \twoheadrightarrow A/\gamma$ the canonical projection. The following equality is immediate:

$$p_{\theta,\gamma}^A \circ p_\theta^A = p_\gamma^A, \quad \text{for all } \theta \supseteq \gamma \text{ in } S_A. \tag{4.1}$$

Given $\theta$ in $S_A$, we denote by $\pi_\theta^A\colon A \twoheadrightarrow A/\theta$ the canonical projection. The following equality holds

$$p_\theta^A \circ t_A = \pi_\theta^A, \quad \text{for all } \theta \in S_A. \tag{4.2}$$

**Claim.** *Let* $f\colon A \to B$ *be a morphism in* $\mathcal{M}_n^{\mathrm{b}\dagger}$. *Let* $\beta \in S_B$. *The following statement are satisfied*

(1) $f^{-1}(\beta) \in S_A$.
(2) *If the map* $A/f^{-1}(\beta) \to B/\beta$ *induced by* $f$ *does not preserve bounds, then* $\beta \in Q_B$ *and* $A/f^{-1}(\beta)$ *is the two-element chain.*

*Proof of Claim.* Denote by $g\colon A/f^{-1}(\beta) \hookrightarrow B/\beta$ the morphism induced by $f$. Notice that $g$ is a morphism in $\mathcal{M}_n^{\mathrm{b}\dagger}$.

If $\beta \supseteq \alpha_B$, then $B/\beta$ is distributive, hence $A/f^{-1}(\beta)$ is distributive. It follows from Remark 4.17 that $g$ is a $0,1$-homomorphism. Moreover $f^{-1}(\beta) \supseteq \alpha_A$, therefore $f^{-1}(\beta) \in S_A$.

Assume that $\beta \nsupseteq \alpha_B$, it follows that $\beta \in Q_B$. If $f^{-1}(\beta) \nsupseteq \alpha_A$, then $A/f^{-1}(\beta)$ is not distributive, however $A/f^{-1}(\beta)$ embeds into $B/\beta \in \{M_k \mid 3 \leq k < \omega\}$, therefore $A/f^{-1}(\beta)$ is simple. It follows that $f^{-1}(\beta) \in Q_A$.

Assume that $\beta \nsupseteq \alpha_B$ and $A/f^{-1}(\beta)$ is not a two-element chain. Notice that $A/f^{-1}(\beta)$ has at least three elements. As $B/\beta \in \{M_k \mid 3 \leq k < \omega\}$ and $g$ is an embedding, it follows that $A/f^{-1}(\beta)$ belongs (up to isomorphism) to $\{M_k \mid 3 \leq k < \omega\} \cup \{\mathbf{3}, \mathbf{2}^2\}$.

As the length of $B/\beta$ is two and the length of all lattices in $\{M_k \mid 3 \leq k < \omega\} \cup \{\mathbf{3}, \mathbf{2}^2\}$ is two, it follows that $g$ is a $0,1$-homomorphism. □ Claim.

Let $f\colon A \to B$. Let $\beta \in S_B$. If $A/f^{-1}(\beta)$ is not a two-element chain or $\beta \notin Q_B$, we denote by $f_\beta\colon A/f^{-1}(\beta) \to B/\beta$ the morphism induced by $f$. It follows from the claim that $f$ is a $0,1$-homomorphism.

If $A/f^{-1}(\beta)$ is a two-element chain and $\beta \in Q_B$, we denote by $f_\beta\colon A/f^{-1}(\beta) \to B/\beta$ the only $0,1$-homomorphism.

Let $\gamma \supseteq \theta \supseteq \alpha_B$ in $\operatorname{Con} B$. Notice that $\theta, \gamma \notin Q_B$, hence $f_\theta \colon A/f^{-1}(\theta) \to B/\theta$ is the map induced by $f$ and $f_\gamma \colon A/f^{-1}(\gamma) \to B/\gamma$ is the map induced by $f$. Thus the following equality holds

$$p^B_{\theta,\gamma} \circ f_\theta = f_\gamma \circ p^A_{f^{-1}(\theta),f^{-1}(\gamma)}, \quad \text{for all } \gamma \supseteq \theta \supseteq \alpha_B \text{ in } \operatorname{Con} B. \tag{4.3}$$

We denote

$$\begin{aligned} \Psi(f) \colon \Psi(A) &\to \Psi(B) \\ u &\mapsto (f_\beta(p^A_{f^{-1}(\beta)}(u)))_{\beta \in R_B} \end{aligned}$$

Notice that

$$p^B_\theta \circ \Psi(f) = f_\theta \circ p^A_{f^{-1}(\theta)}, \quad \text{for each } \theta \in Q_B. \tag{4.4}$$

Let $\theta \supseteq \alpha_B$, the following equalities hold

$$\begin{aligned} p^B_\theta \circ \Psi(f) &= p^B_{\alpha_B,\theta} \circ p^B_{\alpha_B} \circ \Psi(f), && \text{by (4.1).} \\ &= p^B_{\alpha_B,\theta} \circ f_{\alpha_B} \circ p^A_{f^{-1}(\alpha_B)}, && \text{by (4.4).} \\ &= f_\theta \circ p^A_{f^{-1}(\alpha_B),f^{-1}(\theta)} \circ p^A_{f^{-1}(\alpha_B)}, && \text{by (4.3).} \\ &= f_\theta \circ p^A_{f^{-1}(\theta)}, && \text{by (4.1).} \end{aligned}$$

It follows that

$$p^B_\theta \circ \Psi(f) = f_\theta \circ p^A_{f^{-1}(\theta)}, \quad \text{for each } \theta \in S_B. \tag{4.5}$$

Let $f \colon A \to B$ and $g \colon B \to C$ be morphisms in $\mathcal{M}^{\mathrm{b}\dagger}_n$. Let $\gamma \in S_C$. Assume that $A/f^{-1}(g^{-1}(\gamma))$ is not a two-element chain. As $A/f^{-1}(g^{-1}(\gamma))$ embeds into $B/g^{-1}(\gamma)$ it follows that $B/g^{-1}(\gamma)$ is not a two-element chain. Thus $f_{g^{-1}(\gamma)} \colon A/f^{-1}(g^{-1}(\gamma)) \to B/g^{-1}(\gamma)$ is the morphism induced by $f$, similarly $g_\gamma$ is induced by $g$, and $(g \circ f)_\gamma$ is induced by $g \circ f$, therefore $(g \circ f)_\gamma = g_\gamma \circ f_{g^{-1}(\gamma)}$.

Assume that $A/f^{-1}(g^{-1}(\gamma))$ is a two-element chain. Notice that both $(g \circ f)_\gamma$ and $g_\gamma \circ f_{g^{-1}(\gamma)}$ preserve bounds, however there is a unique $0,1$-homomorphism $A/f^{-1}(g^{-1}(\gamma)) \to C/\gamma$. Therefore the following equality holds

$$(g \circ f)_\gamma = g_\gamma \circ f_{g^{-1}(\gamma)}, \quad \text{for all } \gamma \in S_C. \tag{4.6}$$

Let $\gamma \in S_C$. The following equalities hold

$$\begin{aligned} p^C_\gamma \circ \Psi(g \circ f) &= (g \circ f)_\gamma \circ p^A_{f^{-1}(g^{-1}(\gamma))}, && \text{by (4.5).} \\ &= g_\gamma \circ f_{g^{-1}(\gamma)} \circ p^A_{f^{-1}(g^{-1}(\gamma))}, && \text{by (4.6).} \\ &= g_\gamma \circ p^B_{g^{-1}(\gamma)} \circ \Psi(f), && \text{by (4.5).} \\ &= p^C_\gamma \circ \Psi(g) \circ \Psi(f), && \text{by (4.5).} \end{aligned}$$

Thus $\Psi(g \circ f) = \Psi(g) \circ \Psi(f)$. Moreover it is easy to prove that $\Psi(\mathrm{id}_A) = \mathrm{id}_{\Psi(A)}$ for all $A$ in $\mathcal{M}^{\mathrm{b}\dagger}_n$. Therefore $\Psi$ is a functor.

Let $f \colon A \to B$ be a morphism in $\mathcal{M}^{\mathrm{b}\dagger}_n$. Denote by $g \colon A/f^{-1}(\gamma) \to B/\gamma$ the morphism induced by $f$. Notice that $g$ is a morphism in $\mathcal{M}^{\mathrm{b}\dagger}_n$. Let $\gamma \in S_C$. The following equalities hold

$$\begin{aligned} p^B_\gamma \circ \Psi(f) \circ t_A &= f_\gamma \circ p^A_{f^{-1}(\gamma)} \circ t_A, && \text{by (4.5).} \\ &= f_\gamma \circ \pi^A_{f^{-1}(\gamma)}, && \text{by (4.2).} \end{aligned}$$

The following equalities hold

$$\begin{aligned} p_\gamma^B \circ t_B \circ f &= \pi_\gamma^B \circ f, && \text{by (4.2).} \\ &= g \circ \pi^A_{f^{-1}(\gamma)}, && \text{as } g \text{ is induced by } f. \end{aligned}$$

If $A/f^{-1}(\gamma)$ is not a two-element chain then the map $f_\gamma$ is induced by $f$, hence $g = f_\gamma$, thus $\operatorname{Con} f_\gamma = \operatorname{Con} g$.

Assume that $A/f^{-1}(\gamma)$ is a two-element chain. Both map $\operatorname{Con} f_\gamma$ and $\operatorname{Con} g$ are $(\vee,0,1)$-homomorphism, however there is only one $(\vee,0,1)$-homomorphism $\operatorname{Con} A/f^{-1}(\gamma) \to \operatorname{Con} B/\gamma$, thus $\operatorname{Con} f_\gamma = \operatorname{Con} g$. Therefore the following equality holds

$$(\operatorname{Con} p_\gamma^B) \circ (\operatorname{Con} \Psi(f)) \circ \xi_A = (\operatorname{Con} p_\gamma^B) \circ \xi_B \circ (\operatorname{Con} f), \quad \text{for all } \gamma \in R_B$$

It implies that $(\operatorname{Con} \Psi(f)) \circ \xi_A = \xi_B \circ (\operatorname{Con} f)$. Thus $(\xi_A \mid A \in \mathcal{M}_n^{\mathrm{b}\dagger})$ is a natural equivalence. □

**Corollary 4.19.** *Let $n \geq 3$. There exists a functor $\Psi\colon \mathcal{M}_n^{\mathrm{b}} \to \mathcal{M}_n^{0,1}$, preserving colimits and such that $\operatorname{Con_c} \circ \Psi$ is naturally equivalent to $\operatorname{Con_c}$.*

*Proof.* Let $\Psi\colon \mathcal{M}_n^{\mathrm{b}\dagger} \to \mathcal{M}_n^{0,1}$ be the functor constructed in Lemma 4.18, such that $\operatorname{Con_c} \circ \Psi$ is naturally equivalent to $\operatorname{Con_c}$.

The following statements hold.

1. (1) The category $\mathcal{M}_n^{\mathrm{b}\dagger}$ is a full subcategory of finitely presented objects in $\mathcal{M}_n^{\mathrm{b}}$ (cf. Lemma 3.3(3)).
2. (2) The category $\mathcal{M}_n^{0,1}$ has all small directed colimits.
3. (3) All objects in $\mathcal{M}_n^{\mathrm{b}}$ is a small directed colimits of objects in $\mathcal{M}_n^{\mathrm{b}\dagger}$ (cf. Corollary 4.11).
4. (4) The category $\mathcal{M}_n^{\mathrm{b}}$ has all small hom-sets.

It follows from [4, Proposition 1-4.2] that there exists a functor $\overline{\Psi}\colon \mathcal{M}_n^{\mathrm{b}} \to \mathcal{M}_n^{0,1}$ such that $\overline{\Psi} \restriction \mathcal{M}_n^{\mathrm{b}\dagger} = \Psi$ and $\overline{\Psi}$ preserves all small directed colimits.

Denote by $\mathcal{S}$ the category of $(\vee,0,1)$-semilattice with $(\vee,0,1)$-homomorphism. The category $\mathcal{S}$ has all small directed colimits. Both functors $\operatorname{Con_c} \circ \overline{\Psi}\colon \mathcal{M}_n^{\mathrm{b}} \to \mathcal{S}$ and $\operatorname{Con_c}\colon \mathcal{M}_n^{\mathrm{b}} \to \mathcal{S}$ preserve all small directed colimits and $\operatorname{Con_c} \circ \overline{\Psi} \restriction \mathcal{M}_n^{\mathrm{b}\dagger} = \operatorname{Con_c} \circ \Psi \cong \operatorname{Con_c} \restriction \mathcal{M}_n^{\mathrm{b}\dagger}$. Therefore it follows from the uniqueness (cf. [4, Remark 1-4.5]) that $\operatorname{Con_c} \circ \overline{\Psi} \cong \operatorname{Con_c}$. □

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