CRITICAL POINTS BETWEEN VARIETIES GENERATED BY SUBSPACE LATTICES OF VECTOR SPACES

PIERRE GILLIBERT

Abstract. We denote by $\mathrm{Con_c}\, A$ the semilattice of all compact congruences of an algebra A. Given a variety $\mathcal{V}$ of algebras, we denote by $\mathrm{Con_c}\,\mathcal{V}$ the class of all semilattices isomorphic to $\mathrm{Con_c}\, A$ for some $A \in \mathcal{V}$. Given varieties $\mathcal{V}$ and $\mathcal{W}$ of algebras, the *critical point* of $\mathcal{V}$ under $\mathcal{W}$ is defined as $\mathrm{crit}(\mathcal{V}; \mathcal{W}) = \min\{\mathrm{card}\, D \mid D \in \mathrm{Con_c}\,\mathcal{V} - \mathrm{Con_c}\,\mathcal{W}\}$. Given a finitely generated variety $\mathcal{V}$ of modular lattices, we obtain an integer ℓ, depending on $\mathcal{V}$, such that $\mathrm{crit}(\mathcal{V}; \mathbf{Var}(\mathrm{Sub}\, F^n)) \geq \aleph_2$ for any $n \geq \ell$ and any field F.

In a second part, using tools introduced in [5], we prove that:

$$\mathrm{crit}(\mathcal{M}_n; \mathbf{Var}(\mathrm{Sub}\, F^3)) = \aleph_2,$$

for any finite field F and any ordinal n such that $2+\mathrm{card}\, F \leq n \leq \omega$. Similarly $\mathrm{crit}(\mathbf{Var}(\mathrm{Sub}\, F^3); \mathbf{Var}(\mathrm{Sub}\, K^3)) = \aleph_2$, for all finite fields F and K such that $\mathrm{card}\, F > \mathrm{card}\, K$.

1. Introduction

We denote by $\mathrm{Con}\, A$ (resp., $\mathrm{Con_c}\, A$) the lattice (resp., $(\vee, 0)$-semilattice) of all congruences (resp., compact congruences) of an algebra A. For a homomorphism $f\colon A \to B$ of algebras, we denote by $\mathrm{Con}\, f$ the map from $\mathrm{Con}\, A$ to $\mathrm{Con}\, B$ defined by the rule

$$(\mathrm{Con}\, f)(\alpha) = \text{congruence of } B \text{ generated by } \{(f(x), f(y)) \mid (x, y) \in \alpha\},$$

for every $\alpha \in \mathrm{Con}\, A$, and we also denote by $\mathrm{Con_c}\, f$ the restriction of $\mathrm{Con}\, f$ from $\mathrm{Con_c}\, A$ to $\mathrm{Con_c}\, B$.

A *congruence-lifting* of a $(\vee, 0)$-semilattice S is an algebra A such that $\mathrm{Con_c}\, A \cong S$. Given a variety $\mathcal{V}$ of algebras, *the compact congruence class of* $\mathcal{V}$, denoted by $\mathrm{Con_c}\,\mathcal{V}$, is the class of all $(\vee, 0)$-semilattices isomorphic to $\mathrm{Con_c}\, A$ for some $A \in \mathcal{V}$. As illustrated by [12], even the compact congruence classes of small varieties of lattices are complicated objects. For example, in case $\mathcal{V}$ is the variety of all lattices, $\mathrm{Con_c}\,\mathcal{V}$ contains all distributive $(\vee, 0)$-semilattices of cardinality at most $\aleph_1$, but not all distributive $(\vee, 0)$-semilattices (cf. [15]).

Given varieties $\mathcal{V}$ and $\mathcal{W}$ of algebras, the *critical point* of $\mathcal{V}$ and $\mathcal{W}$, denoted by $\mathrm{crit}(\mathcal{V}; \mathcal{W})$, is the smallest cardinality of a $(\vee, 0)$-semilattice in $\mathrm{Con_c}(\mathcal{V}) - \mathrm{Con_c}(\mathcal{W})$ if it exists, or ∞, otherwise (i.e., if $\mathrm{Con_c}\,\mathcal{V} \subseteq \mathrm{Con_c}\,\mathcal{W}$).

Date: February 12, 2010.

2000 *Mathematics Subject Classification.* 08A30. Secondary 16E50, 51D25, 06B20.

Key words and phrases. Lattice; semilattice; diagram; lifting; compact; congruence; homomorphism; critical point; norm-covering; supported; support; ideal; Von Neumann regular ring; dimension monoid.

This paper is a part of the author's "Doctorat de l'université de Caen", prepared under the supervision of Friedrich Wehrung.

Let I be a poset. A *direct system indexed by* I is a family $(A_i, f_{i,j})_{i\leq j \text{ in } I}$ such that A_i is an algebra, $f_{i,j}\colon A_i \to A_j$ is a morphism of algebras, $f_{i,i} = \mathrm{id}_{A_i}$, and $f_{i,k} = f_{j,k} \circ f_{i,j}$, for all $i \leq j \leq k$ in I.

Denote by $\operatorname{Sub} V$ the subspace lattice of a vector space V, and by $\mathcal{M}_n$ the variety of lattices generated by the lattice M_n of length two with n atoms, for $3 \leq n \leq \omega$. Using the theory of the *dimension monoid* of a lattice, introduced by F. Wehrung in [13], together with some von Neumann regular ring theory, we prove in Section 3 that if $\mathcal{V}$ is a finitely generated variety of modular lattices with all subdirectly irreducible members of length less or equal to n, then $\operatorname{crit}(\mathcal{V}; \mathbf{Var}(\operatorname{Sub} F^n)) \geq \aleph_2$ for any field F. As an immediate application, $\operatorname{crit}(\mathcal{M}_n; \mathcal{M}_3) \geq \aleph_2$ for every n with $3 \leq n \leq \omega$ (cf. Corollary 3.12). Thus, by using the result of M. Ploščica in [10], we obtain the equality $\operatorname{crit}(\mathcal{M}_m; \mathcal{M}_n) = \aleph_2$ for all m, n with $3 \leq n < m \leq \omega$. Our proof does not rely on the approach used by Ploščica in [11] to prove the inequality $\operatorname{crit}(\mathcal{M}_m^{0,1}; \mathcal{M}_n^{0,1}) \geq \aleph_2$, and it extends that result to the unbounded case. We also obtain a new proof of that result in Section 4, that does not even rely on the approach used by Ploščica in [10] to prove the inequality $\operatorname{crit}(\mathcal{M}_m; \mathcal{M}_n) \leq \aleph_2$.

Let $\mathcal{V}$ be a variety of lattices, let $\vec{D}$ be a diagram of $(\vee, 0)$-semilattices and $(\vee, 0)$-homomorphisms. A *congruence-lifting* of $\vec{D}$ in $\mathcal{V}$ is a diagram $\vec{L}$ of $\mathcal{V}$ such that the composite $\operatorname{Con_c} \circ \vec{L}$ is naturally equivalent to $\vec{D}$.

In Section 4, we give a diagram of finite $(\vee, 0)$-semilattices that is congruence-liftable in $\mathcal{M}_n$, but not congruence-liftable in $\mathbf{Var}(\operatorname{Sub} F^3)$, for any finite field F and any n such that $2+\operatorname{card} F \leq n \leq \omega$. As the diagram of $(\vee, 0)$-semilattices is indexed by some "good" lattice, we obtain, using results of [5], that $\operatorname{crit}(\mathcal{M}_n; \mathbf{Var}(\operatorname{Sub} F^3)) = \aleph_2$. This implies immediately that $\operatorname{crit}(\mathcal{M}_4; \mathcal{M}_{3,3}) = \aleph_2$. Let F and K be finite fields such that $\operatorname{card} F > \operatorname{card} K$, we also obtain $\operatorname{crit}(\mathbf{Var}(\operatorname{Sub} F^3); \mathbf{Var}(\operatorname{Sub} K^3)) = \aleph_2$.

In a similar way, we prove that $\operatorname{crit}(\mathcal{M}_\omega; \mathcal{V}) = \aleph_2$, for every finitely generated variety of lattices $\mathcal{V}$ such that $M_3 \in \mathcal{V}$.

2. Basic concepts

We denote by $\operatorname{dom} f$ the domain of any function f. A *poset* is a partially ordered set. Given a poset P, we put

$$Q \downarrow X = \{p \in Q \mid (\exists x \in X)(p \leq x)\}, \qquad Q \uparrow X = \{p \in Q \mid (\exists x \in X)(p \geq x)\},$$

for any $X, Q \subseteq P$, and we will write $\downarrow X$ (resp., $\uparrow X$) instead of $P \downarrow X$ (resp., $P \uparrow X$) in case P is understood. We shall also write $\downarrow p$ instead of $\downarrow \{p\}$, and so on, for $p \in P$. A poset P is *lower finite* if $P \downarrow p$ is finite for all $p \in P$. For $p, q \in P$ let $p \prec q$ hold, if $p < q$ and there is no $r \in P$ with $p < r < q$, in this case p is called a *lower cover* of q. We denote by $P^{=}$ the set of all non-maximal elements in a poset P. We denote by $\mathrm{M}(L)$ the set of all completely meet-irreducible elements of a lattice L.

A 2-*ladder* is a lower finite lattice in which every element has at most two lower covers. S. Z. Ditor constructs in [1] a 2-ladder of cardinality $\aleph_1$.

For a set X and a cardinal κ, we denote by:

$$\begin{aligned} [X]^{\kappa} &= \{Y \subseteq X \mid \operatorname{card} Y = \kappa\}, \\ [X]^{\leq\kappa} &= \{Y \subseteq X \mid \operatorname{card} Y \leq \kappa\}, \\ [X]^{<\kappa} &= \{Y \subseteq X \mid \operatorname{card} Y < \kappa\}. \end{aligned}$$

Denote by $\mathscr{P}$ the category with objects the ordered pairs (G, u) where G is a pre-ordered abelian group and u is an order-unit of G (i.e., for each $x \in G$, there exists an integer n with $-nu \leq x \leq nu$), and morphisms $f\colon (G,u) \to (H,v)$ where $f\colon G \to H$ is an order-preserving group homomorphism and $f(u) = v$.

We denote by Dim the functor that maps a lattice to its *dimension monoid*, introduced by F. Wehrung in [13], we also denote by $\Delta(a,b)$ for $a \leq b$ in L the canonical generators of $\operatorname{Dim} L$. We denote by K_0^ℓ the functor that maps a lattice to the pre-ordered abelian universal group (also called Grothendieck group) of its dimension monoid. If L is a bounded lattice then (the canonical image in $K_0^\ell(L)$ of) $\Delta(0_L, 1_L)$ is an order-unit of $K_0^\ell(L)$. If $f\colon L \to L'$ is a $0,1$-preserving homomorphism of bounded lattices, then $K_0^\ell(f)\colon (K_0^\ell(L), \Delta(0_L,1_L)) \to (K_0^\ell(L'), \Delta(0_{L'}, 1_{L'}))$ preserves the order-unit.

All our rings are associative but not necessarily unital.

- We denote by $\mathbb{L}(R)$ the poset of principal right ideals of every regular ring R. The results of Fryer and Halperin in [4, Section 3.2], imply that, $\mathbb{L}(R)$ is a 0-lattice, and for any homomorphism $f\colon R \to S$ of regular rings, the map $\mathbb{L}(f)\colon \mathbb{L}(R) \to \mathbb{L}(S)$, $I \mapsto f(I)S$ is a 0-lattice homomorphism (cf. Micol's thesis [9, Theorem 1.4] for the unital case). Hence $\mathbb{L}$ is a functor from the category of regular rings to the category of 0-lattices with 0-lattice homomorphisms.
- We denote by V the functor from the category of unital rings with morphisms preserving units to the category of commutative monoids, that maps a unital ring R to the commutative monoid of all isomorphism classes of finitely generated projective right R-modules and any homomorphism $f\colon R \to S$ of unital rings to the monoid homomorphism $V(f)\colon V(R) \to V(S)$, $\sum_i e_i R \mapsto \sum_i f(e_i)S$.

We denote by $\operatorname{Id} R$ (resp., $\operatorname{Id_c} R$) the lattice of all two-sided ideals (resp., finitely generated two-sided ideals) of any ring R. We denote by $\operatorname{Sub} E$ the subspace lattice of a vector space E. We denote by $\mathrm{M}_n(F)$ the F-algebra of $n \times n$ matrices with entries from F, for every field F and every positive integer n. A *matricial F-algebra* is an F-algebra of the form $\mathrm{M}_{k_1}(F) \times \cdots \times \mathrm{M}_{k_n}(F)$, for positive integers $k_1, \dots, k_n$.

For a finitely generated projective right module P over a unital ring R, we denote by $[P]$ the corresponding element in $K_0(R)$, that is, the stable isomorphism class of P. We refer to [7, Section 15] for the required notions about the K_0 functor.

A *K_0-lifting* of a pre-ordered abelian group with order-unit (G,u) is a regular ring R such that $(K_0(R), [R]) \cong (G,u)$. A K_0-lifting of a diagram $\vec{G}\colon I \to \mathscr{P}$ is a diagram $\vec{R}\colon I \to \mathscr{P}$ such that $(K_0(-), [-]) \circ \vec{R} \cong \vec{G}$.

We denote by ∇ the functor that sends a monoid to it maximal semilattice quotient, that is, $\nabla(M) = M/{\asymp}$ where $\asymp$ is the smallest congruence of M such that $M/{\asymp}$ is a semilattice. We denote by $\overline{\nabla}$ the functor that maps a partially pre-ordered abelian group G to $\nabla(G^+)$ where G^+ is the monoid of all positive elements of G.

We denote by $\mathbf{Var}(L)$ (resp., $\mathbf{Var}_0(L)$, resp., $\mathbf{Var}_{0,1}(L)$) the variety of lattices (resp., lattices with 0, resp., bounded lattices) generated by a lattice L.

A lattice K is a *congruence-preserving extension of* a lattice L, if L is a sublattice of K and $\operatorname{Con_c} i\colon \operatorname{Con} L \to \operatorname{Con} K$ is an isomorphism, where $i\colon L \to K$ is the inclusion map.

We denote by M_n and $M_{n,m}$ the lattices represented in Figure 1, for $3 \leq m, n \leq \omega$, and by $\mathcal{M}_n$ and $\mathcal{M}_{n,m}$, respectively, the lattice varieties that they generate. We also denote by $\mathcal{M}_n^0$ the variety of lattices with 0 generated by M_n, and so on.

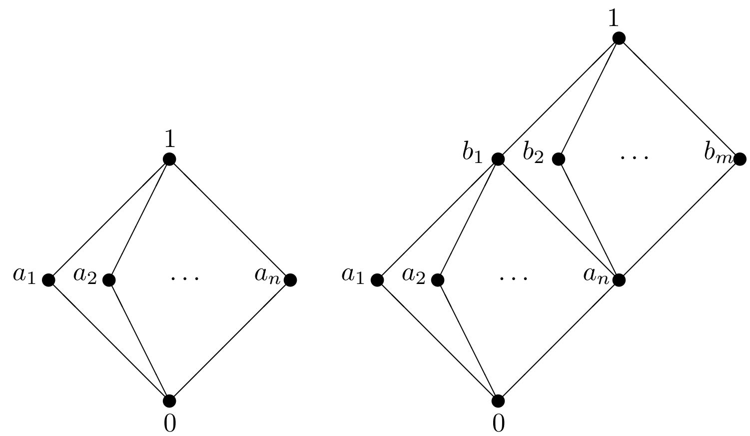

FIGURE 1. The lattices M_n and $M_{n,m}$.

A lattice L *satisfies Whitman's condition* if for all a, b, c, and d in L:

$$a \wedge b \leq c \vee d \quad \text{implies either } a \leq c \vee d \quad \text{or} \quad b \leq c \vee d \quad \text{or} \quad a \wedge b \leq c \quad \text{or} \quad a \wedge b \leq d.$$

The lattice M_n satisfies Whitman's condition for all $n \geq 3$.

3. Lower bounds for some critical points

The following proposition is proved in [13, Proposition 5.5].

Proposition 3.1. *Let L be a modular lattice without infinite bounded chains. Let P be the set of all projectivity classes of prime intervals of L. Given $\xi \in P$, denote by $|a, b|_\xi$ the number of occurrences of an interval in ξ in any maximal chain of the interval $[a, b]$. Then there exists an isomorphism $\pi \colon \operatorname{Dim} L \to (\mathbb{Z}^+)^{(P)}$ such that $\pi(\Delta(a, b)) = (|a, b|_\xi \mid \xi \in P)$ for all $a \leq b$ in L.*

This makes it possible to prove the following lemma, which gives an explicit description of $K_0^\ell(L)$ for every modular lattice L of finite length (in such a case the set P is finite).

Lemma 3.2. *Let L be a modular lattice of finite length, set $X = \mathrm{M}(\operatorname{Con} L)$. Then there exists an isomorphism $\pi' \colon K_0^\ell(L) \to \mathbb{Z}^X$ such that*

$$\pi'(\Delta(a, b)) = (\mathrm{lh}([a/\theta, b/\theta]) \mid \theta \in X), \quad \textit{for all } a \leq b \textit{ in } L.$$

In particular $(K_0^\ell(L), \Delta(0, 1))$ is isomorphic to $(\mathbb{Z}^X, (\mathrm{lh}(L/\theta))_{\theta \in X})$.

Proof. Denote by P be the set of all projectivity classes of prime intervals of L. For any $\xi \in P$ denote by θ_ξ the largest congruence of L that does not collapse any prime intervals in ξ. As L is modular of finite length, the congruences of L are in one-to-one correspondence with subsets of P (cf. [6, Chapter III]), and so the assignment $\xi \mapsto \theta_\xi$ defines a bijection from P onto X. Moreover any prime interval not in ξ is collapsed by θ_ξ, for any $\xi \in P$. Let $a \leq b$ in L, let $\xi \in P$. Let $a_0 \prec a_1 \prec \cdots \prec a_n$ in L such that $a_0 = a$ and $a_n = b$. Let $0 \leq r_1 < r_2 < \cdots < r_s < n$ be all the integers such that $[a_{r_k}, a_{r_k+1}] \in \xi$ for all $1 \leq k \leq s$. Thus $|a, b|_\xi = s$. Set $r_{s+1} = n$.

As $[a_{r_k}, a_{r_k+1}] \in \xi$ and $[a_{r_k+t}, a_{r_k+t+1}] \notin \xi$ for all $1 \leq t \leq r_{k+1} - r_k - 1$, we obtain that

$$a_{r_k}/\theta_\xi \prec a_{r_k+1}/\theta_\xi = a_{r_k+2}/\theta_\xi = \cdots = a_{r_{k+1}}/\theta_\xi, \quad \text{for all } 1 \leq k \leq s.$$

Thus the following covering relations hold:

$$a/\theta_\xi = a_{r_1}/\theta_\xi \prec a_{r_2}/\theta_\xi \prec \cdots \prec a_{r_s}/\theta_\xi \prec a_{r_{s+1}}/\theta_\xi = b/\theta_\xi.$$

So $\mathrm{lh}([a/\theta_\xi, b/\theta_\xi]) = s = |a, b|_\xi$. We conclude the proof by using Proposition 3.1. $\square$

Proposition 3.3. *The following natural equivalences hold*

$$\begin{array}{llll} (i) & \nabla \circ \mathrm{Dim} \cong \mathrm{Con_c} & & \textit{on lattices} \\ (ii) & \nabla \circ V \cong \mathrm{Con_c} \circ \mathbb{L} & & \textit{on regular rings} \end{array}$$

Proof. (i) follows from [13, Corollary 2.3], while (ii) is contained in [7, Corollary 2.23]; see also the proof of [14, Proposition 4.6]. $\square$

We shall always apply this result to unital regular rings R such that $V(R)$ is cancellative (i.e., R is unit-regular), so $K_0(R)^+ = V(R)$, and to lattices L such that $\mathrm{Dim}\, L$ is cancellative, so $K_0^\ell(L)^+ \cong \mathrm{Dim}\, L$. Here G^+ denotes the positive cone of G, for any partially pre-ordered abelian group G.

The following theorem is proved in [7, Theorem 15.23].

Theorem 3.4. *Let F be a field, let R be a matricial F-algebra, and let S be a unit-regular F-algebra.*

1. *Given any morphism $f\colon (K_0(R), [R]) \to (K_0(S), [S])$ in $\mathscr{P}$, the category of pre-ordered abelian groups with order-unit (cf. Section 2), there exists an F-algebra homomorphism $\phi\colon R \to S$ such that $K_0(\phi) = f$.*
2. *If $\phi, \psi\colon R \to S$ are F-algebra homomorphisms, then $K_0(\phi) = K_0(\psi)$ if and only if there exists an inner automorphism θ of S such that $\phi = \theta \circ \psi$.*

The following lemma is folklore.

Lemma 3.5. *Let F be a field, let $\boldsymbol{u} = (u_k)_{1\leq k\leq n}$ be a family of positive integers, let $R = \prod_{k=1}^{n} \mathrm{M}_{u_k}(F)$. Then $(K_0(R), [R]) \cong (\mathbb{Z}^n, \boldsymbol{u})$.*

Lemma 3.6. *Let F be a field. Let I be a 2-ladder, let $G_i = (\mathbb{Z}^{n_i}, \boldsymbol{u}^i = (u^i_k)_{1\leq k\leq n_i})$ such that $\boldsymbol{u}^i$ is an order-unit, let $R_i = \prod_{k=1}^{n_i} \mathrm{M}_{u^i_k}(F)$ for all $i \in I$. Let $f_{i,j}\colon G_i \to G_j$ for all $i \leq j$ in I such that $\vec{G} = (G_i, f_{i,j})_{i\leq j \text{ in } I}$ is a direct system in $\mathscr{P}$. Then there exists a direct system $(R_i, \phi_{i,j})_{i\leq j \text{ in } I}$ of matricial F-algebra which is a K_0-lifting of $(G_i, f_{i,j})_{i\leq j \text{ in } I}$.*

Proof. By Lemma 3.5 there exists an isomorphism $\tau_i\colon (K_0(R_i), [R_i]) \to G_i = (\mathbb{Z}^{n_i}, \boldsymbol{u}^i)$ in $\mathscr{P}$, for all $i \in I$. Let $g_{i,j} = \tau_j^{-1} \circ f_{i,j} \circ \tau_i$, for all $i \leq j$ in I.

For $i = j = 0$ (the smallest element of I), we put $\phi_{0,0} = \mathrm{id}_{R_0}$. Let $i \in I$ with a lower cover i'. It follows from Theorem 3.4(1) that there exists $\psi_{i',i}\colon R_{i'} \to R_i$ such that $K_0(\psi_{i',i}) = g_{i',i}$.

If i has only i' as lower cover, assume that we have a direct system $(R_j, \phi_{j,k})_{j\leq k\leq i'}$ lifting $(G_j, f_{j,k})_{j\leq k\leq i'}$. Set $\phi_{j,i} = \psi_{i',i} \circ \phi_{j,i'}$ for all $j < i$, and $\phi_{i,i} = \mathrm{id}_{R_i}$. It is easy to see that $(R_i, \phi_{j,k})_{j\leq k\leq i}$ is a direct system lifting $(G_j, f_{j,k})_{j\leq k\leq i}$.

Let i has two distinct lower covers i' and i'', and set $\ell = i' \wedge i''$. Assume that we have direct system $(R_j, \phi_{j,k})_{j\leq k\leq i'}$ and $(R_j, \phi_{j,k})_{j\leq k\leq i''}$ lifting $(G_j, f_{j,k})_{j\leq k\leq i'}$ and $(G_j, f_{j,k})_{j\leq k\leq i''}$ respectively. The following equalities hold

$$K_0(\psi_{i',i} \circ \phi_{\ell,i'}) = K_0(\psi_{i',i}) \circ K_0(\phi_{\ell,i'}) = g_{i',i} \circ g_{\ell,i'} = g_{\ell,i}$$

Similarly $K_0(\psi_{i'',i} \circ \phi_{\ell,i''}) = g_{\ell,i} = K_0(\psi_{i',i} \circ \phi_{\ell,i'})$, thus, by Theorem 3.4(2), there exists an inner automorphism θ of R_i such that $\theta \circ \psi_{i'',i} \circ \phi_{\ell,i''} = \psi_{i',i} \circ \phi_{\ell,i'}$. Put $\phi_{i',i} = \psi_{i',i}$ and $\phi_{i'',i} = \theta \circ \psi_{i'',i}$. Thus $\phi_{i',i} \circ \phi_{i'\wedge i'',i'} = \phi_{i'',i} \circ \phi_{i'\wedge i'',i''}$, so we can construct a direct system $(R_j, \phi_{j,k})_{j\leq k\leq i}$.

Hence, by induction, we obtain a direct system $(R_i, \phi_{i,j})_{i\leq j \text{ in } I}$ of matricial F-algebras, such that $K_0(\phi_{i,j}) = g_{i,j}$ for all $i \leq j$ in I as required. □

Lemma 3.7. *Let F be a field. Let L be a bounded modular lattice such that all finitely generated sublattices of L have finite length. Assume that* $\operatorname{card} L \leq \aleph_1$. *Then there exists a locally matricial ring R such that* $\operatorname{Con} L \cong \operatorname{Con} \mathbb{L}(R)$ *and* $\mathbb{L}(R) \in \mathbf{Var}_{0,1}(\operatorname{Sub} F^n \mid n < \omega)$.

Moreover if there exists $n < \omega$ such that $n \geq \operatorname{lh}(K)$ for each simple lattice $K \in \mathbf{Var}(L)$ of finite length, then there exists a locally matricial ring R such that $\operatorname{Con} L \cong \operatorname{Con} \mathbb{L}(R)$ *and* $\mathbb{L}(R) \in \mathbf{Var}_{0,1}(\operatorname{Sub} F^n)$.

Proof. Let I be a 2-ladder of cardinality $\aleph_1$. Pick a surjection $\rho\colon I \twoheadrightarrow L$ and denote by L_i the sublattice of L generated by $\rho(I \downarrow i) \cup \{0, 1\}$, for each $i \in I$. Furthermore, denote by $f_{i,j}\colon L_i \to L_j$ the inclusion map, for all $i \leq j$ in I. Then $\vec{L} = (L_i, f_{i,j})_{i\leq j \text{ in } I}$ is a direct system of modular lattices of finite length and $0,1$-lattice embeddings.

Assume that there exists $n < \omega$ such that $n \geq \operatorname{lh}(K)$ for each simple lattice $K \in \mathbf{Var}(L)$ of finite length. Let $\vec{G} = K_0^\ell \circ \vec{L}$, set $X_i = \mathrm{M}(\operatorname{Con} L_i)$ for all $i \in I$, and set $r_x^i = \operatorname{lh}(L_i/x)$ for each $x \in X_i$. The congruence lattice of any modular lattice of finite length is Boolean (cf. [6, Chapter III]), in particular, every subdirectly irreducible modular lattice of finite length is simple. This applies to the subdirectly irreducible lattice L_i/x, which is therefore simple. Thus $r_x^i \leq n$, for all $i \in I$ and all $x \in X_i$. By Lemma 3.2, $G_i \cong (\mathbb{Z}^{X_i}, (r_x^i)_{x\in X_i})$ for all $i \in I$.

Set $R_i = \prod_{x\in X_i} \mathrm{M}_{r_x^i}(F)$. By Lemma 3.5, $(K_0(R_i), [R_i]) \cong (\mathbb{Z}^{X_i}, (r_x^i)_{x\in X}) \cong G_i$. By Lemma 3.6, there exists a direct system $\vec{R} = (R_i, \phi_{i,j})_{i\leq j \text{ in } I}$ with morphisms preserving units, such that:

$$K_0 \circ \vec{R} \cong \vec{G} = K_0^\ell \circ \vec{L}. \tag{3.1}$$

Moreover:

$$\mathbb{L}(R_i) \cong \mathbb{L}\left(\prod_{x\in X_i} \mathrm{M}_{r_x^i}(F)\right) \cong \prod_{x\in X_i} \mathbb{L}(\mathrm{M}_{r_x^i}(F)) \cong \prod_{x\in X_i} \operatorname{Sub} F^{r_x^i} \in \mathbf{Var}_{0,1}(\operatorname{Sub} F^n).$$

Let $R = \varinjlim \vec{R}$. As $\mathbb{L}$ preserves direct limits, $\mathbb{L}(R) \cong \varinjlim(\mathbb{L} \circ \vec{R})$, but $\mathbb{L} \circ \vec{R}$ is a diagram of $\mathbf{Var}_{0,1}(\operatorname{Sub} F^n)$, so $\mathbb{L}(R) \in \mathbf{Var}_{0,1}(\operatorname{Sub} F^n)$. Moreover the following isomorphisms hold:

$$\begin{aligned}
\operatorname{Con_c} \mathbb{L}(R) &\cong \overline{\nabla}(K_0(R)) && \text{by Proposition 3.3}\\
&\cong \overline{\nabla}(K_0(\varinjlim \vec{R})) &&\\
&\cong \overline{\nabla}(\varinjlim(K_0 \circ \vec{R})) && \text{as } K_0 \text{ preserves direct limits}\\
&\cong \overline{\nabla}(\varinjlim(K_0^\ell \circ \vec{L})) && \text{by (3.1)}\\
&\cong \overline{\nabla}(K_0^\ell(\varinjlim \vec{L})) && \text{as } K_0^\ell \text{ preserves direct limits}\\
&\cong \overline{\nabla}(K_0^\ell(L)) &&\\
&\cong \operatorname{Con_c} L && \text{by Proposition 3.3.}
\end{aligned}$$

The other case, without restriction on finite lengths of simple lattices, is similar. □

Lemma 3.7 works for bounded lattices, however any lattice can be embedded into a bounded lattice. In the rest of this section, using this result, we extend Lemma 3.7 to unbounded lattices.

Lemma 3.8. *Let L be a lattice, let $L' = L \sqcup \{0,1\}$ such that 0 is the smallest element of L' and 1 is the largest. Let $f\colon L \hookrightarrow L'$ be the inclusion map. Then $\operatorname{Con_c} f$ is a injective $(\vee, 0)$-homomorphism and $(\operatorname{Con_c} f)(\operatorname{Con_c} L)$ is an ideal of $\operatorname{Con_c} L'$.*

Proof. Let $\theta \in \operatorname{Con_c} L$, let $L'_\theta = (L/\theta) \sqcup \{0,1\}$ such that 0 is the smallest element of L'_θ and 1 is its largest element. The following map

$$g\colon L' \to L'_\theta$$
$$x \mapsto \begin{cases} 0 & \text{if } x = 0\\ 1 & \text{if } x = 1\\ x/\theta & \text{if } x \in L\end{cases}$$

is a lattice homomorphism, and $\ker g = \theta \cup \{(0,0),(1,1)\}$, so the latter is a congruence of L'. It follows that $(\operatorname{Con_c} f)(\theta) = \theta \cup \{(0,0),(1,1)\}$. Thus $\operatorname{Con_c} f$ is an embedding. Let $\beta = \bigvee_{i=1}^n \Theta_{L'}(x_i,y_i) \in \operatorname{Con_c} L'$, such that $\beta \subseteq (\operatorname{Con_c} f)(\theta)$. We can assume that $x_i \neq y_i$ for all $1 \leq i \leq n$. Thus, as $(x_i,y_i) \in \theta \cup \{(0,0),(1,1)\}$, $(x_i,y_i) \in \theta$ for all $1 \leq i \leq n$. Let $\alpha = \bigvee_{i=1}^n \Theta_L(x_i,y_i)$, then $(\operatorname{Con_c} f)(\alpha) = \beta$. Thus $(\operatorname{Con_c} f)(\operatorname{Con_c} L)$ is an ideal of $\operatorname{Con_c} L'$. □

F. Wehrung proves the following proposition in [14, Corollary 4.4]; the result also applies to the non-unital case, with a similar proof.

Proposition 3.9. *For any regular ring R, $\operatorname{Con_c} \mathbb{L}(R)$ is isomorphic to $\operatorname{Id_c} R$.*

Lemma 3.10. *Let R be a regular ring, and let I be a two-sided ideal of R. Then the following assertions hold*

1. *The set I is a regular subring of R.*
2. *Any right (resp., left) ideal of I is a right (resp., left) ideal of R.*
3. *In particular $\operatorname{Id}(I) = \operatorname{Id}(R) \downarrow I$, and $\mathbb{L}(I) = \mathbb{L}(R) \downarrow I$.*

Proof. The assertion (1) follows from [7, Lemma 1.3].

Let J be a right ideal of I, let $a \in J$, let $x \in R$. As I is regular there exists $y \in I$ such that $a = aya$, so $ax = ayax$, but $a \in I$, so $yax \in I$, moreover J is a right ideal of I, so $ax = ayax \in J$. Thus J is a right ideal of R. Similarly any left ideal of I is a left ideal of R. Thus $\operatorname{Id}(I) = \operatorname{Id}(R) \downarrow I$.

Let $a \in R$ idempotent. If $aR \subseteq I$, then $a \in I$, so $aI \subseteq aR = aaR \subseteq aI$, and so $aI = aR$, thus $aR \in \mathbb{L}(I)$. So $\mathbb{L}(I) = \mathbb{L}(R) \downarrow I$. $\square$

Theorem 3.11. *Let F be a field. Let $\mathcal{V}$ be a variety of modular lattices (resp., a variety of bounded modular lattices). Assume that all finitely generated lattices of $\mathcal{V}$ have finite length. Then*

$$\operatorname{crit}(\mathcal{V}; \mathbf{Var}_0(\operatorname{Sub} F^n \mid n \in \omega)) \geq \aleph_2 \quad (\textit{resp., } \operatorname{crit}(\mathcal{V}; \mathbf{Var}_{0,1}(\operatorname{Sub} F^n \mid n \in \omega)) \geq \aleph_2).$$

Moreover for $L \in \mathcal{V}$ of cardinality at most $\aleph_1$, there exists a regular ring A such that $\operatorname{Con} L \cong \operatorname{Con} \mathbb{L}(A)$ and $\mathbb{L}(A) \in \mathbf{Var}_0(\operatorname{Sub} F^n \mid n \in \omega)$ (resp., $\mathbb{L}(A) \in \mathbf{Var}_{0,1}(\operatorname{Sub} F^n \mid n \in \omega)$).

If there exists $n < \omega$ such that $\operatorname{lh}(K) \leq n$ for each simple lattice $K \in \mathcal{V}$ of finite length, then:

$$\operatorname{crit}(\mathcal{V}; \mathbf{Var}_0(\operatorname{Sub} F^n)) \geq \aleph_2 \quad (\textit{resp., } \operatorname{crit}(\mathcal{V}; \mathbf{Var}_{0,1}(\operatorname{Sub} F^n)) \geq \aleph_2).$$

Moreover for $L \in \mathcal{V}$ of cardinality at most $\aleph_1$, there exists a regular ring A such that $\operatorname{Con} L \cong \operatorname{Con} \mathbb{L}(A)$ and $\mathbb{L}(A) \in \mathbf{Var}_0(\operatorname{Sub} F^n)$ (resp., $\mathbb{L}(A) \in \mathbf{Var}_{0,1}(\operatorname{Sub} F^n)$).

Observe that $\mathbb{L}(A)$ is, in addition, relatively complemented; in particular, it is congruence-permutable.

Proof. The bounded case is an immediate application of Lemma 3.7.

Let $\mathcal{V}$ be a variety of modular lattices in which finitely generated lattices have finite length. Let $L \in \mathcal{V}$ such that $\operatorname{card} L \leq \aleph_1$, let $L' = L \sqcup \{0, 1\}$ as in Lemma 3.8 and let D be the ideal of $\operatorname{Con_c} L'$ corresponding to $\operatorname{Con_c} L$. By Chapter I, Section 4, Exercise 14 in [6] we have $L' \in \mathcal{V}$, thus, by Lemma 3.7, there exists a regular ring R such that $\mathbb{L}(R) \in \mathbf{Var}_0(\operatorname{Sub} F^n)$, and $\operatorname{Con_c} \mathbb{L}(R) \cong \operatorname{Con_c} L'$. By Proposition 3.9, $\operatorname{Con_c} \mathbb{L}(R) \cong \operatorname{Id_c} R$. Let I be the ideal of R corresponding to D. Then $\operatorname{Con} L \cong \operatorname{Id} D \cong \operatorname{Id} R \downarrow I \cong \operatorname{Id} I \cong \operatorname{Con} \mathbb{L}(I)$. Moreover $\mathbb{L}(I) = \mathbb{L}(R) \downarrow I$ belongs to $\mathcal{W}$. $\square$

We obtain the following generalization of M. Ploščica's results in [11].

Corollary 3.12. *Let m, n be ordinals such that $3 \leq n < m \leq \omega$. Then the equality $\operatorname{crit}(\mathcal{M}_m; \mathcal{M}_n) = \aleph_2$ holds.*

Proof. Every simple lattice of $\mathcal{M}_n$ has length at most two. Moreover, $\operatorname{Sub} \mathbb{F}_2^2 \cong M_3 \in \mathcal{M}_n$, where $\mathbb{F}_2$ is the two-element field. Thus, by Theorem 3.11, $\operatorname{crit}(\mathcal{M}_m; \mathcal{M}_n) \geq \aleph_2$.

Conversely, M. Ploščica proves in [10] that there exists a $(\vee, 0)$-semilattice of cardinality $\aleph_2$, congruence-liftable in $\mathcal{M}_m$, but not congruence-liftable in $\mathcal{M}_n$. So $\operatorname{crit}(\mathcal{M}_m; \mathcal{M}_n) \leq \aleph_2$. $\square$

In Section 4 we shall give another $(\vee, 0)$-semilattice of cardinality $\aleph_2$, congruence-liftable in $\mathcal{M}_m$, but not congruence-liftable in $\mathcal{M}_n$.

4. An upper bound of some critical points

Using the results of [5], we first prove that if a simple lattice of a variety of lattices $\mathcal{V}$ has larger length than all simple lattices of a finitely generated variety of lattices $\mathcal{W}$, then $\operatorname{crit}(\mathcal{V}; \mathcal{W}) \leq \aleph_0$.

Remark 4.1. Let $x \prec y$ in a lattice L. Let $(\alpha_i)_{i\in I}$ be a family of congruences of L, if $(x, y) \in \bigvee_{i\in I} \alpha_i$, then $(x, y) \in \alpha_i$ for some $i \in I$. In particular there exists a largest congruence separating x and y. Such a congruence is completely meet-irreducible, and in a lattice of finite height all completely meet-irreducible congruences are of this form.

Lemma 4.2. *Let L be a lattice and let $n \geq 0$. If $\operatorname{Con_c} L \cong 2^n$ then $\operatorname{lh}(L) \geq n$. Moreover, if C is a finite maximal chain of L, then $\operatorname{Con_c} f$ is surjective, where $f : C \to L$ is the inclusion map.*

Proof. If L has no finite maximal chain then $\operatorname{lh}(L) \geq n$ is immediate. Assume that C is a finite maximal chain of L. Denotes by $0 = x_0 \prec x_1 \prec \cdots \prec x_m = 1$ the elements of C. Denote by $f : C \to L$ the inclusion map.

Let $k \in \{0, \ldots, m-1\}$. We have $x_k \prec x_{k+1}$, hence $\Theta_L(x_k, x_{k+1})$ is join-irreducible in $\operatorname{Con_c} L$. As $\operatorname{Con_c} L$ is Boolean, $\Theta_L(x_k, x_{k+1})$ is an atom of $\operatorname{Con_c} L$.

Let θ be an atom of $\operatorname{Con_c} L$, we have:

$$\theta \leq \Theta_L(0, 1) = \bigvee_{k=0}^{m-1} \Theta_L(x_k, x_{k+1})$$

So there exists $k \in \{0, \ldots, m-1\}$ such that $\theta \leq \Theta_L(x_k, x_{k+1})$. As $\Theta_L(x_k, x_{k+1})$ is an atom of $\operatorname{Con_c} L$, we have $\theta = \Theta_L(x_k, x_{k+1})$. It follows that $\operatorname{Con_c} f$ is surjective, so $m \geq n$ and so $\operatorname{lh}(L) \geq n$. □

Theorem 4.3. *Let $\mathcal{V}$ be a variety of lattices (resp., a variety of bounded lattices), let $\mathcal{W}$ be a finitely generated variety of lattices, let D be a finite $(\vee, 0)$-semilattice. If there exists a lifting $K \in \mathcal{V}$ of D of length greater than every lifting of D in $\mathcal{W}$, then* $\operatorname{crit}(\mathcal{V}; \mathcal{W}) \leq \aleph_0$*. Moreover if $\mathcal{V}$ is a finitely generated variety of modular lattices and $\mathcal{W}$ is not trivial, then* $\operatorname{crit}(\mathcal{V}; \mathcal{W}) = \aleph_0$*.*

Proof. As D is finite, taking a sublattice, we can assume that $\operatorname{card} K \leq \aleph_0$. Let n be the greatest length of a lifting of D in $\mathcal{W}$. As $\operatorname{lh}(K) > n$, there exists a chain C of K of length $n+1$ (resp., we can assume that C has 0 and 1). Let $f : C \to K$ be the inclusion map. Assume that there exists a lifting $g : C' \to K'$ of $\operatorname{Con_c} f$ in $\mathcal{W}$. As f is an embedding, g is also an embedding. As $\operatorname{Con_c} K' \cong \operatorname{Con_c} K \cong D$, $\operatorname{lh}(K') \leq n$. Moreover $\operatorname{Con_c} C' \cong \operatorname{Con_c} C \cong 2^{n+1}$, thus, by Lemma 4.2, $\operatorname{lh}(C') = n+1$. So $n \geq \operatorname{lh}(K') \geq \operatorname{lh}(C') = n+1$; a contradiction.

Therefore $\operatorname{Con_c} f$ has no lifting in $\mathcal{W}$. So, as $\operatorname{card} K \leq \aleph_0$ and by [5, Corollary 7.6], $\operatorname{crit}(\mathcal{V}; \mathcal{W}) \leq \aleph_0$ (in the bounded case f preserves bounds, thus the result of [5] also applies).

Moreover if $\mathcal{V}$ is a finitely generated variety of modular lattices, then the finite $(\vee, 0)$-semilattices with congruence-lifting in $\mathcal{V}$ are the finite Boolean lattices. Finite Boolean lattices are also liftable in $\mathcal{W}$. Hence $\operatorname{crit}(\mathcal{V}; \mathcal{W}) = \aleph_0$. □

The following corollary is an immediate application of Theorem 4.3 and Theorem 3.11. It shows that the critical point between a finitely generated variety of modular lattices and a variety generated by a lattice of subspaces of a finite vector space, cannot be $\aleph_1$.

Corollary 4.4. *Let $\mathcal{V}$ be a finitely generated variety of modular lattices, let F be a finite field, let $n \geq 1$ be an integer. If there exists a simple lattice in $K \in \mathcal{V}$ such that $\operatorname{lh}(K) > n$, then $\operatorname{crit}(\mathcal{V}; \mathbf{Var}(\operatorname{Sub} F^n)) = \aleph_0$, else $\operatorname{crit}(\mathcal{V}; \mathbf{Var}(\operatorname{Sub} F^n)) \geq \aleph_2$.*

We shall now give a diagram of $(\vee, 0)$-semilattices $\vec{S}$, congruence-liftable in $\mathcal{M}_n$, such that for every finitely generated variety $\mathcal{V}$, generated by lattices of length at most three, the diagram $\vec{S}$ is congruence-liftable in $\mathcal{V}$ if and only if $M_n \in \mathcal{V}$.

Let $n \geq 3$ be an integer. Set $\underline{n} = \{0, 1, \dots, n-1\}$, and set:

$$I_n = \{P \in \mathfrak{P}(\underline{n}) \mid \text{either } \operatorname{card}(P) \leq 2 \text{ or } P = \underline{n}\}.$$

Denote by $a_0, \dots, a_{n-1}$ the atoms of M_n. Set $A_P = \{a_x \mid x \in P\} \cup \{0, 1\}$, for all $P \in I_n$. Let $f_{P,Q} \colon A_P \to A_Q$ be the inclusion map for all $P \subseteq Q$ in I_n. Then $\vec{A} = (A_P, f_{P,Q})_{P \subseteq Q \text{ in } I_n}$ is a direct system in $\mathcal{M}_n^{0,1}$. The diagram $\vec{S}$ is defined as $\operatorname{Con_c} \circ \vec{A}$.

Lemma 4.5. *Let $\vec{B} = (B_P, g_{P,Q})_{P \subseteq Q \text{ in } I_n}$ be a congruence-lifting of $\operatorname{Con_c} \circ \vec{A}$ by lattices, with all the maps $g_{P,Q}$ inclusion maps, for all $P \subseteq Q$ in I_n. Let $u < v$ in $B_\emptyset$. Let $P \in I_n$ then:*

$$\Theta_{B_P}(u, v) = B_P \times B_P, \quad \text{the largest congruence of } B_P.$$

Let $\vec{\xi} = (\xi_P)_{P \in I_n} \colon \operatorname{Con_c} \circ \vec{A} \to \operatorname{Con_c} \circ \vec{B}$ be a natural equivalence. Let $x, y \in \underline{n}$ distinct. Let $b_x \in [u, v]_{B_{\{x\}}}$ and $b_y \in [u, v]_{B_{\{y\}}}$. Set $P = \{x, y\}$. Let $c \in \{0, 1\}$. Then the following assertions hold:

1. *If $\Theta_{B_{\{x\}}}(u, b_x) = \xi_{\{x\}}(\Theta_{A_{\{x\}}}(c, a_x))$, then $\Theta_{B_P}(u, b_x) = \xi_P(\Theta_{A_P}(c, a_x))$.*
2. *If $\Theta_{B_{\{z\}}}(u, b_z) = \xi_{\{z\}}(\Theta_{A_{\{z\}}}(c, a_z))$ for all $z \in \{x, y\}$, then $b_x \wedge b_y = u$.*
3. *If $\Theta_{B_{\{x\}}}(b_x, v) = \xi_{\{x\}}(\Theta_{A_{\{x\}}}(c, a_x))$, then $\Theta_{B_P}(b_x, v) = \xi_P(\Theta_{A_P}(c, a_x))$.*
4. *If $\Theta_{B_{\{z\}}}(b_z, v) = \xi_{\{z\}}(\Theta_{A_{\{z\}}}(c, a_z))$ for all $z \in \{x, y\}$, then $b_x \vee b_y = v$.*
5. *If $\Theta_{B_{\{x\}}}(u, b_x) = \xi_{\{x\}}(\Theta_{A_{\{x\}}}(c, a_x))$ and $\Theta_{B_{\{y\}}}(b_y, v) = \xi_{\{y\}}(\Theta_{A_{\{y\}}}(c, a_y))$, then $b_x \leq b_y$.*

Proof. As $f_{P,Q}$ preserves bounds, $\operatorname{Con_c} f_{P,Q}$ preserves bounds, thus $\operatorname{Con_c} g_{P,Q}$ preserves bounds, for all $P \subseteq Q$ in I_n. Let $u < v$ in $B_\emptyset$. As $B_\emptyset$ is simple, $\Theta_{B_\emptyset}(u, v)$ is the largest congruence of $B_\emptyset$. Moreover, $\operatorname{Con_c} g_{\emptyset,P}$ preserves bounds, for all $P \in I_n$. Hence:

$$\Theta_{B_P}(u, v) = B_P \times B_P, \quad \text{the largest congruence of } B_P.$$

(1) The following equalities hold:

$$\begin{aligned}
\Theta_{B_P}(u, b_x) &= (\operatorname{Con_c} g_{\{x\},P})(\Theta_{B_{\{x\}}}(u, b_x)) \\
&= (\operatorname{Con_c} g_{\{x\},P})(\xi_{\{x\}}(\Theta_{A_{\{x\}}}(c, a_x))) && \text{by assumption} \\
&= \xi_P \circ (\operatorname{Con_c} f_{\{x\},P})(\Theta_{A_{\{x\}}}(c, a_x)) \\
&= \xi_P(\Theta_{A_P}(c, a_x)).
\end{aligned}$$

(2) The following containments hold:

$$\begin{aligned}
\Theta_{B_P}(u, b_x \wedge b_y) &\subseteq \Theta_{B_P}(u, b_x) \cap \Theta_{B_P}(u, b_y) \\
&= \xi_P(\Theta_{A_P}(c, a_x)) \cap \xi_P(\Theta_{A_P}(c, a_y)) && \text{by (1)} \\
&= \xi_P(\Theta_{A_P}(c, a_x) \cap \Theta_{A_P}(c, a_y)) \\
&= \xi_P(\mathrm{id}_{A_P}) = \mathrm{id}_{B_P}.
\end{aligned}$$

so $u = b_x \wedge b_y$.

(3) Similar to (1).

(4) Similar to (2).

(5) The following containments hold:

$$\begin{aligned}\Theta_{B_P}(b_y, b_x \vee b_y) &\subseteq \Theta_{B_P}(u, b_x) \cap \Theta_{B_P}(b_y, v)\\ &= \xi_P(\Theta_{A_P}(c, a_x)) \cap \xi_P(\Theta_{A_P}(c, a_y)) &&\text{by (1) and (3)}\\ &= \xi_P(\Theta_{A_P}(c, a_x) \cap \Theta_{A_P}(c, a_y))\\ &= \xi_P(\mathrm{id}_{A_P}) = \mathrm{id}_{B_P}.\end{aligned}$$

so $b_y = b_x \vee b_y$, thus $b_x \leq b_y$. □

The following lemma shows that if we have some "small" enough congruence-lifting of $\mathrm{Con_c} \circ \vec{A}$ in a variety, then M_n belongs to this variety.

Lemma 4.6. *Let $\vec{B} = (B_P, g_{P,Q})_{P\subseteq Q \text{ in } I_n}$ be a congruence-lifting of $\mathrm{Con_c} \circ \vec{A}$ by lattices. Assume that $B_{\{x\}}$ is a chain of length two for all $x \in \underline{n}$. Then M_n can be embedded into $B_{\underline{n}}$.*

Proof. Let $\vec{\xi} = (\xi_P)_{P\in I_n} \colon \mathrm{Con_c} \circ \vec{A} \to \mathrm{Con_c} \circ \vec{B}$ be a natural equivalence. As $f_{P,Q}$ is an embedding, $\mathrm{Con_c}\, f_{P,Q}$ separates 0, so $\mathrm{Con_c}\, g_{P,Q}$ separates 0, hence $g_{P,Q}$ is an embedding, thus we can assume that $g_{P,Q}$ is the inclusion map from B_P into B_Q, for all $P \subseteq Q$ in I_n.

Let $u < v$ in $B_\emptyset$. By Lemma 4.5, $\Theta_{B_{\{x\}}}(u, v)$ is the largest congruence of $B_{\{x\}}$. Moreover $B_{\{x\}}$ is the 3-element chain, so u is the smallest element of $B_{\{x\}}$ while v is its largest element. Denote by b_x the middle element of $B_{\{x\}}$.

The congruence $\xi_{\{x\}}(\Theta_{A_{\{x\}}}(0, a_x))$ is join-irreducible, thus it is either equal to $\Theta_{B_{\{x\}}}(u, b_x)$ or to $\Theta_{B_{\{x\}}}(b_x, v)$. Set:

$$X' = \{x \in \underline{n} \mid \xi_{\{x\}}(\Theta_{A_{\{x\}}}(0, a_x)) = \Theta_{B_{\{x\}}}(u, b_x)\},$$

$$X'' = \{x \in \underline{n} \mid \xi_{\{x\}}(\Theta_{A_{\{x\}}}(0, a_x)) = \Theta_{B_{\{x\}}}(b_x, v)\}.$$

As $\Theta_{A_{\{x\}}}(0, a_x)$ is the complement of $\Theta_{A_{\{x\}}}(a_x, 1)$ and $\Theta_{B_{\{x\}}}(u, b_x)$ is the complement of $\Theta_{B_{\{x\}}}(b_x, v)$, we also get that:

$$X' = \{x \in \underline{n} \mid \xi_{\{x\}}(\Theta_{A_{\{x\}}}(a_x, 1)) = \Theta_{B_{\{x\}}}(b_x, v)\}$$

$$X'' = \{x \in \underline{n} \mid \xi_{\{x\}}(\Theta_{A_{\{x\}}}(a_x, 1)) = \Theta_{B_{\{x\}}}(u, b_x)\}.$$

Moreover $\underline{n} = X' \cup X''$. As $\operatorname{card} \underline{n} \geq 3$, either $\operatorname{card} X' \geq 2$ or $\operatorname{card} X'' \geq 2$.

Assume that $\operatorname{card} X' \geq 2$. Let x, y in X' distinct. By Lemma 4.5(2), $b_x \wedge b_y = u$. By Lemma 4.5(4), $b_x \vee b_y = v$.

Now assume that $X'' \neq \emptyset$. Let $z \in X''$. As $\xi_{\{x\}}(\Theta_{A_{\{x\}}}(0, a_x)) = \Theta_{B_{\{x\}}}(u, b_x)$ and $\xi_{\{z\}}(\Theta_{A_{\{z\}}}(0, a_z)) = \Theta_{B_{\{z\}}}(b_z, v)$, it follows from Lemma 4.5(5) that $b_x \leq b_z$. Similarly, as $\xi_{\{z\}}(\Theta_{A_{\{z\}}}(a_z, 1)) = \Theta_{B_{\{z\}}}(u, b_z)$ and $\xi_{\{y\}}(\Theta_{A_{\{y\}}}(a_y, 1)) = \Theta_{B_{\{y\}}}(b_y, v)$, it follows from Lemma 4.5(5) that $b_z \leq b_y$. Thus $b_x \leq b_y$. So $u = b_x \wedge b_y = b_x > u$, a contradiction.

Thus $X'' = \emptyset$, so $X' = \underline{n}$, and so $\{u, b_0, b_1, \dots, b_n, v\}$ is a sublattice of $B_{\underline{n}}$ isomorphic to M_n. The case $\operatorname{card} X'' \geq 2$ is similar. □

We shall now use a tool introduced in [5] to prove that having a congruence-lifting of $\mathrm{Con_c} \circ \vec{A}$ is equivalent to having a congruence-lifting of some $(\vee, 0)$-semilattice of cardinality $\aleph_2$. This requires the following infinite combinatorial property, proved by A. Hajnal and A. Máté in [8], see also [3, Theorem 46.2]. This property is also used by M. Ploščica in [10].

Proposition 4.7. *Let $n \geq 0$ be an integer, let α be an ordinal, let $\kappa \geq \aleph_{\alpha+2}$, let $f\colon [\kappa]^2 \to [\kappa]^{<\aleph_\alpha}$. Then there exists $Y \in [\kappa]^n$ such that $a \notin f(\{b,c\})$ for all distinct $a, b, c \in Y$.*

Now recall the definition of supported poset and norm-covering introduced in [5, Section 4].

Definition 4.8. A finite subset V of a poset U is a *kernel*, if for every $u \in U$, there exists a largest element $v \in V$ such that $v \leq u$. We denote this element by $V \cdot u$.

We say that U is *supported*, if every finite subset of U is contained in a kernel of U.

We denote by $V \cdot \boldsymbol{u}$ the largest element of $V \cap \boldsymbol{u}$, for every kernel V of U and every ideal $\boldsymbol{u}$ of U. As an immediate application of the finiteness of kernels, we obtain that any intersection of a nonempty set of kernels of a poset U is a kernel of U.

Definition 4.9. A *norm-covering* of a poset I is a pair $(U, |\cdot|)$, where U is a supported poset and $|\cdot|\colon U \to I$, $u \mapsto |u|$ is an order-preserving map.

A *sharp ideal* of $(U, |\cdot|)$ is an ideal $\boldsymbol{u}$ of U such that $\{|v| \mid v \in \boldsymbol{u}\}$ has a largest element. For example, for every $u \in U$, the principal ideal $U \downarrow u$ is sharp; we shall often identify u and $U \downarrow u$. We denote this element by $|\boldsymbol{u}|$. We denote by $\mathrm{Id_s}(U, |\cdot|)$ the set of all sharp ideals of $(U, |\cdot|)$, partially ordered by inclusion.

A sharp ideal $\boldsymbol{u}$ of $(U, |\cdot|)$ is *extreme*, if there is no sharp ideal $\boldsymbol{v}$ with $\boldsymbol{v} > \boldsymbol{u}$ and $|\boldsymbol{v}| = |\boldsymbol{u}|$. We denote by $\mathrm{Id_e}(U, |\cdot|)$ the set of all extreme ideals of $(U, |\cdot|)$.

Let κ be a cardinal number. We say that $(U, |\cdot|)$ is *κ-compatible*, if for every order-preserving map $F\colon \mathrm{Id_e}(U, |\cdot|) \to \mathfrak{P}(U)$ such that $\operatorname{card} F(\boldsymbol{u}) < \kappa$ for all $\boldsymbol{u} \in \mathrm{Id_e}(U, |\cdot|)^{=}$, there exists an order-preserving map $\sigma\colon I \to \mathrm{Id_e}(U, |\cdot|)$ such that:

(1) The equality $|\sigma(i)| = i$ holds for all $i \in I$.
(2) The containment $F(\sigma(i)) \cap \sigma(j) \subseteq \sigma(i)$ holds for all $i \leq j$ in I.

Lemma 4.10. *Let X be a set, let $(A_x)_{x \in X}$ be a family of sets, let:*

$$U = \bigsqcup_{P \in [X]^{<\omega}} \prod_{x \in P} A_x.$$

We view the elements of U as (partial) functions and "to be greater" means "to extend". Then U is a supported poset.

Proof. Let V be a finite subset of U. Let $Y_x = \{u_x \mid u \in V \text{ and } x \in \operatorname{dom} u\}$ for all $x \in X$. Let $D = \bigcup_{u \in V} \operatorname{dom} u$. Let:

$$W = \{u \in U \mid \operatorname{dom} u \subseteq D \text{ and } (\forall x \in \operatorname{dom} u)(u_x \in Y_x)\}$$

the set D, and the sets Y_x for $x \in X$ are all finite, so W is finite.

Let $u \in U$, let $P = \{x \in \operatorname{dom} u \mid x \in D \text{ and } u_x \in Y_x\}$. Then $u \restriction P \in W$. Moreover let $w \in W$ such that $w \leq u$. Let $x \in \operatorname{dom} w$, then $x \in D$, and $u_x = w_x \in Y_x$, thus $\operatorname{dom} w \subseteq P$, so $w \leq u \restriction P$. Therefore $u \restriction P$ is the largest element of $W \downarrow u$. $\square$

Using Lemma 4.10 and Proposition 4.7 we can construct a $\aleph_\alpha$-compatible lower finite norm-covering of I_n, the poset constructed earlier.

Lemma 4.11. *Let α be an ordinal. Let $U = \bigsqcup_{P \in \mathfrak{P}(\underline{n})} \aleph_{\alpha+2}^{P}$, partially ordered by inclusion. Let*

$$\begin{aligned} |\cdot| \colon U &\to I_n \\ u &\mapsto |u| = \begin{cases} \operatorname{dom} u & \text{if } \operatorname{card}(\operatorname{dom} u) \leq 2 \\ \underline{n} & \text{otherwise.} \end{cases} \end{aligned}$$

Then $(U, |\cdot|)$ is a $\aleph_\alpha$-compatible lower finite norm-covering of I_n. Moreover $\operatorname{card} U = \aleph_{\alpha+2}$.

Proof. By Lemma 4.10, the set U is supported. Moreover $|\cdot|$ preserves order, so $(U, |\cdot|)$ is a norm-covering of I_n. The poset U is lower finite.

Extreme ideals are of the form $\downarrow u$, where $u \in U$ and $\operatorname{dom} u \in I_n$, so we identify the corresponding extreme ideal with u. Thus $\operatorname{Id}_{\mathrm{e}}(U, |\cdot|) = \{u \in U \mid \operatorname{dom} u \in I_n\}$.

Let $F \colon \operatorname{Id}_{\mathrm{e}}(U, |\cdot|) \to \mathfrak{P}(U)$ be an order-preserving map such that $\operatorname{card} F(\boldsymbol{u}) < \aleph_\alpha$ for all $\boldsymbol{u} \in \operatorname{Id}_{\mathrm{e}}(U, |\cdot|)^{=}$, let

$$\begin{aligned} G \colon [\aleph_{\alpha+2}]^2 &\to [\aleph_{\alpha+2}]^{<\aleph_\alpha} \\ s &\mapsto \bigcup \left\{ \operatorname{im} v \mid u \in \bigcup_{P \in I_n - \{\underline{n}\}} s^P \text{ and } v \in F(u) \right\}. \end{aligned}$$

By Proposition 4.7, there exists $A \subset \aleph_{\alpha+2}$ such that $\operatorname{card} A = n$ and $a \notin G(\{b, c\})$ for all distinct $a, b, c \in A$. Let $u \colon \underline{n} \to A$ be a one-to-one map. Let $\phi \colon I_n \to \operatorname{Id}_{\mathrm{e}}(U, |\cdot|)$, $P \mapsto u \restriction P$. Then $|\phi(P)| = P$. Let $P \subsetneq Q$ in I_n, let $v \in F(u \restriction P) \downarrow (u \restriction Q)$. Let $x \in \operatorname{dom} v - P$. As $P \in I_n$, and $P \neq \underline{n}$, $\operatorname{card} P \leq 2$. Let $P' = \{y, z\} \subseteq \underline{n}$, such that y, z are distinct, $P \subseteq P'$, and $x \notin P'$. Let $s = \{u_y, u_z\}$, then $u \restriction P' \in s^{P'}$, as $v \in F(u \restriction P) \subseteq F(u \restriction P')$, $v_x \in G(s)$. Moreover $u_x, u_y, u_z \in A$ are distinct, thus $u_x \notin G(\{u_y, u_z\}) = G(s)$, so $v_x \neq u_x$ in contradiction with $v \leq u$, so $\operatorname{dom} v \subseteq P$, and so $v \leq u \restriction P$. □

Using the results of [5] together with Lemma 4.11, we obtain the following result.

Lemma 4.12. *Let $\mathcal{V}$ be a variety of algebras with a countable similarity type, let $\mathcal{W}$ be a finitely generated congruence-distributive variety such that* $\operatorname{crit}(\mathcal{V}; \mathcal{W}) > \aleph_2$. *Let $\vec{D} \colon I_n \to \mathcal{S}$ be a diagram of finite $(\vee, 0)$-semilattices. If $\vec{D}$ is congruence-liftable in $\mathcal{V}$, then $\vec{D}$ is congruence-liftable in $\mathcal{W}$.*

Proof. In this proof we use, but do not give, many definitions of [5]. By Lemma 4.11 there exists $(U, |\cdot|)$ a $\aleph_0$-compatible lower finite norm-covering of I_n such that $\operatorname{card} U = \aleph_2$. Let J be a one-element ordered set. By [5, Lemma 3.9], $\mathcal{W}$ is $(\operatorname{Id}_{\mathrm{e}}(U, |\cdot|)^{=}, J, \aleph_0)$-Löwenheim-Skolem.

Let $\vec{A} = (A_P, f_{P,Q})_{P \subseteq Q \text{ in } I_n}$ be a congruence-lifting of $\vec{D}$ in $\mathcal{V}$. As $\operatorname{Con_c} A_P$ is finite, using [5, Lemma 3.6], taking sublattices we can assume that A_P is countable for all $P \in I_n$. By [5, Lemma 6.7], there exists an U-quasi-lifting $(\tau, \operatorname{Cond}(\vec{A}, U))$ of $\vec{D}$ in $\mathcal{V}$. Moreover:

$$\operatorname{card} \operatorname{Cond}(\vec{A}, U) \leq \sum_{V \in [U]^{<\omega}} \operatorname{card} \left(\prod_{u \in V} A_{|u|} \right) \leq \sum_{V \in [U]^{<\omega}} \aleph_0 \leq \aleph_2$$

As $\operatorname{crit}(\mathcal{V};\mathcal{W}) > \aleph_2$, there are $B \in \mathcal{W}$ and an isomorphism $\xi\colon \operatorname{Con_c}\operatorname{Cond}(\vec{A},U) \to \operatorname{Con_c} B$. So $(\tau\circ\xi^{-1}, B)$ is an U-quasi-lifting of $\vec{D}$. Moreover $\mathcal{W}$ is $(\operatorname{Id_e}(U,|\cdot|)^{=}, J, \aleph_0)$-Löwenheim-Skolem, hence, by [5, Theorem 6.9], with $I = I_n$, there exists a congruence-lifting of $\vec{D}$ in $\mathcal{W}$. □

A similar proof, using Lemma 3.6, Lemma 3.7, Lemma 6.7, and Theorem 6.9 in [5] together with Lemma 4.11, yields the following generalization of Lemma 4.12.

Lemma 4.13. *Let $\alpha \geq 1$ be an ordinal. Let $\mathcal{V}$ and $\mathcal{W}$ be varieties of algebras, with similarity types of cardinality $< \aleph_\alpha$. Let $\vec{D} = (D_P, \varphi_{P,Q})_{P\subseteq Q \text{ in } I_n}$ be a direct system of $(\vee,0)$-semilattices. Assume that the following conditions hold:*

(1) $\operatorname{crit}(\mathcal{V};\mathcal{W}) > \aleph_{\alpha+2}$.
(2) $\operatorname{card}(D_P) < \aleph_\alpha$, *for all* $P \in I_n - \{\underline{n}\}$.
(3) $\operatorname{card}(D_{\underline{n}}) \leq \aleph_{\alpha+2}$.
(4) $\vec{D}$ *is congruence-liftable in* $\mathcal{V}$.

Then $\vec{D}$ is congruence-liftable in $\mathcal{W}$.

The following lemma implies, in particular, that a modular lattice of length three is a congruence-preserving extension of one of its subchains.

Lemma 4.14. *Let L be a lattice of length at most three, let u, v in L such that $\Theta_L(u,v) = L \times L$. If $\operatorname{Con_c} L \cong 2^2$, then there exists $x \in L$ with $u < x < v$ such that L is a congruence-preserving extension of the chain $C = \{u, x, v\}$.*

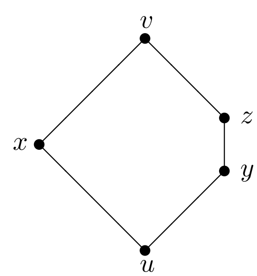

FIGURE 2. The lattice N_5.

Proof. As $\operatorname{Con_c} L \cong 2^2$, $\operatorname{lh}([u,v]) \geq 2$. If $\operatorname{lh}([u,v]) = 2$, then let $C = \{u, x, v\}$, where x is any element such that $u < x < v$. Let $i\colon C \to L$ the inclusion map. The morphism $\operatorname{Con_c} i\colon \operatorname{Con_c} C \to \operatorname{Con_c} L$ is onto, moreover $\operatorname{Con_c} C \cong 2^2 \cong \operatorname{Con_c} L$, so $\operatorname{Con_c} i$ is an isomorphism.

Now assume that $[u,v]$ has length three. As $\operatorname{lh}(L) \leq 3$, $\operatorname{lh}(L) = 3$, u is the smallest element of L, and v is the largest element.

Assume that L has a sublattice isomorphic to N_5, as labeled in Figure 2. Then $C = \{u, y, z, v\}$ is a maximal chain of L. Let $i\colon C \to L$ be the inclusion map. By Lemma 4.2, $\operatorname{Con_c} i$ is surjective. Thus, as $\operatorname{Con} L \cong 2^2$, and $\Theta_L(u,y)$, $\Theta_L(y,z)$, and $\Theta_L(z,v)$ are all the atoms of $\operatorname{Con} L$,

$$\Theta_L(y,z) \subseteq \Theta_L(u,y) \cap \Theta_L(y,z) \cap \Theta_L(z,v) = \operatorname{id}_L,$$

a contradiction. Thus L does not contain any lattice isomorphic to N_5, that is, L is modular.

As $\operatorname{Con} L \cong 2^2$ and $\operatorname{lh}(L) = 3$, L is not distributive. Hence there exists a sublattice of L isomorphic to M_3, let $a < x_1, x_2, x_3 < b$ be its elements. As L is modular, $[a, x_1]_L \cong [x_1, b]_L$, thus $\operatorname{lh}([a, b]_L)$ is even. But $2 \leq \operatorname{lh}([a, b]_L) \leq 3$, so $\operatorname{lh}([a, b]_L) = 2$, thus $a \prec x_1 \prec b$. This chain can be completed into a maximal chain $c \prec a \prec x_1 \prec b$ or $a \prec x_1 \prec b \prec c$. By symmetry, we may assume that $b < c$. Observe that $a = u$ and $c = v$. Set $C = \{u, b, v\}$ and $C_1 = \{u, x_1, b, v\}$. Let $i \colon C \to L$ and $i_1 \colon C_1 \to L$ be the inclusion maps. As C_1 is a maximal chain, $\operatorname{Con_c} i_1$ is onto. As $\Theta_L(u, x_1) = \Theta_L(x_1, b) = \Theta_L(u, b)$, $\operatorname{Con_c} i_1$ and $\operatorname{Con_c} i$ have the same image, thus $\operatorname{Con_c} i$ is onto, so $\operatorname{Con_c} i$ is an isomorphism. $\square$

The result of Lemma 4.14 does not extend to length four or more. The lattice of Figure 3 is not a congruence-preserving extension of any chain with extremities u and v.

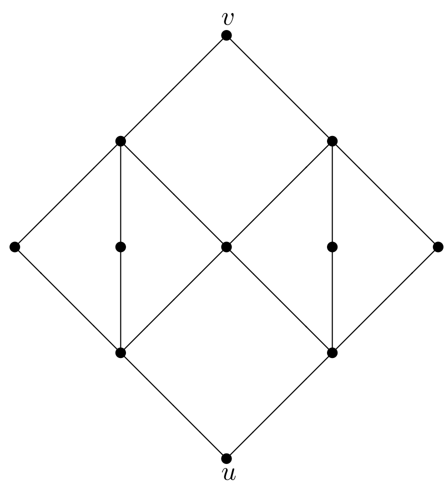

FIGURE 3. Lemma 4.14 does not extend to lattices of greater length.

Lemma 4.15. *Let $n \geq 4$ be an integer, let $\mathcal{V}$ be a finitely generated variety of lattices such that $M_n \notin \mathcal{V}$. If $\operatorname{lh}(K) \leq 3$ for each simple lattice K of $\mathcal{V}$, then* $\operatorname{crit}(\mathcal{M}_n^{0,1}; \mathcal{V}) \leq \aleph_2$.

Proof. We consider the diagram $\vec{A}$ introduced just before Lemma 4.5. Assume that $\operatorname{crit}(\mathcal{M}_n^{0,1}; \mathcal{V}) > \aleph_2$. As $M_n \in \mathcal{M}_n^{0,1}$, $\vec{A}$ is a diagram of $\mathcal{M}_n^{0,1}$ indexed by I_n. By Lemma 4.12, the diagram $\operatorname{Con_c} \circ \vec{A}$ has a congruence-lifting $\vec{B} = (B_P, g_{P,Q})_{P \subseteq Q \text{ in } I_n}$ in $\mathcal{V}$. As $\operatorname{Con} B_{\underline{n}} \cong 2$, the lattice $B_{\underline{n}}$ is simple, thus, by assumption on $\mathcal{V}$, $\operatorname{lh}(B_{\underline{n}}) \leq 3$, and so $\operatorname{lh}(B_{\{x\}}) \leq 3$, for all $x \in \underline{n}$. The lattice $B_\emptyset$ is simple, so, taking a sublattice, we can assume that $B_\emptyset = \{u, v\}$, with $u < v$. By Lemma 4.14, we can assume that $B_{\{x\}}$ is a chain of length two, for each $x \in \underline{n}$. So by Lemma 4.6, M_n is a sublattice of $B_{\underline{n}}$, and so $M_n \in \mathcal{V}$, a contradiction. $\square$

Theorem 4.16. *Let $\mathcal{V}$ be a finitely generated variety of modular lattices and $\mathcal{W}$ be finitely generated variety of lattices. Let $n \geq 3$ be an integer such that $M_n \in \mathcal{V} - \mathcal{W}$. If $\operatorname{lh}(K) \leq 3$ for each simple $K \in \mathcal{V}$, then $\operatorname{crit}(\mathcal{V}; \mathcal{W}) \leq \aleph_2$. Moreover if either $\operatorname{lh}(K) \leq 2$ for each simple $K \in \mathcal{V}$ and $M_3 \in \mathcal{W}$ or $\operatorname{lh}(K) \leq 3$ for each simple $K \in \mathcal{V}$ and $\operatorname{Sub} F^3 \in \mathcal{W}$ for some field F, then $\operatorname{crit}(\mathcal{V}; \mathcal{W}) = \aleph_2$.*

Proof. By Lemma 4.15, $\operatorname{crit}(\mathcal{V};\mathcal{W}) \leq \aleph_2$.

Assume that $\operatorname{lh}(K) \leq 2$ for each simple $K \in \mathcal{V}$ and $M_3 \in \mathcal{W}$. As $\operatorname{Sub}\mathbb{F}_2^2 \cong M_3 \in \mathcal{W}$, it follows from Theorem 3.11 that $\operatorname{crit}(\mathcal{V};\mathcal{W}) \geq \aleph_2$.

Assume that $\operatorname{lh}(K) \leq 3$ for each simple $K \in \mathcal{V}$ and $\operatorname{Sub} F^3 \in \mathcal{W}$ for some field F, it follows from Theorem 3.11 that $\operatorname{crit}(\mathcal{V};\mathcal{W}) \geq \aleph_2$. □

Similarly we obtain the following critical points.

Corollary 4.17. *The following equalities hold*

$$\begin{aligned}
\operatorname{crit}(\mathcal{M}_n;\mathcal{M}_{m,m}) &= \aleph_2;\\
\operatorname{crit}(\mathcal{M}_n^{0,1};\mathcal{M}_{m,m}) &= \aleph_2;\\
\operatorname{crit}(\mathcal{M}_n^{0,1};\mathcal{M}_{m,m}^{0,1}) &= \aleph_2;\\
\operatorname{crit}(\mathcal{M}_n;\mathcal{M}_{m,m}^{0}) &= \aleph_2;\\
\operatorname{crit}(\mathcal{M}_n;\mathcal{M}_{m}^{0}) &= \aleph_2, \qquad \textit{for all } n,m \textit{ with } 3 \leq m < n \leq \omega.
\end{aligned}$$

Proof. Let $n' \leq n$ be an integer such that $m < n' < \omega$. As $M_{n'} \notin \mathcal{M}_{m,m}$, it follows from Lemma 4.15 that $\operatorname{crit}(\mathcal{M}_{n'}^{0,1};\mathcal{M}_{m,m}) \leq \aleph_2$, thus:

$$\operatorname{crit}(\mathcal{M}_n^{0,1};\mathcal{M}_{m,m}) \leq \aleph_2. \tag{4.1}$$

Moreover $M_3 \in \mathcal{M}_{m,m}$, simple lattices of $\mathcal{M}_{m,m}$ are of length at most 3, and finitely generated lattices of $\mathcal{M}_n$ have finite length (and are even finite). Thus, by Theorem 3.11

$$\operatorname{crit}(\mathcal{M}_n;\mathcal{M}_{m,m}^{0}) \geq \aleph_2. \tag{4.2}$$

Similarly:

$$\operatorname{crit}(\mathcal{M}_n^{0,1};\mathcal{M}_{m,m}^{0,1}) \geq \aleph_2. \tag{4.3}$$

All the desired equalities are immediate consequences of (4.1), (4.2), and (4.3). □

As an immediate consequence we obtain:

Corollary 4.18. $\operatorname{crit}(\mathcal{M}_{4,3};\mathcal{M}_{3,3}) \leq \aleph_2$.

This question was suggested by M. Ploščica.

Lemma 4.19. *Let F be field. Then $M_n \in \mathbf{Var}(\operatorname{Sub} F^3)$ if and only if $n \leq 1 + \operatorname{card} F$.*

Proof. If F is infinite then the result is obvious. So we can assume that F is finite.

If $n \leq 1+\operatorname{card} F$, then M_n is a sublattice of $M_{1+\operatorname{card} F} \cong \operatorname{Sub} F^2 \in \mathbf{Var}(\operatorname{Sub} F^3)$, thus $M_n \in \mathbf{Var}(\operatorname{Sub} F^3)$.

Now assume that $M_n \in \mathbf{Var}(\operatorname{Sub} F^3)$. By Jónsson's Lemma, M_n is a homomorphic image of a sublattice of $\operatorname{Sub} F^3$. As M_n satisfies Whitman's condition, it follows from the Davey-Sands Theorem [2, Theorem 1] that M_n is projective in the class of all finite lattices. Therefore, as $\operatorname{Sub} F^3$ is finite, M_n is a sublattice of $\operatorname{Sub} F^3$. Thus there exist distinct subspaces $A, B, V_1, V_2, \dots, V_n$ of F^3 such that $V_i \cap V_j = A$ and $V_i + V_j = B$, for all $1 \leq i < j \leq n$. Let i,j,k distinct. Then:

$$\dim V_i + \dim V_j = \dim B + \dim A = \dim V_i + \dim V_k.$$

Thus $\dim V_j = \dim V_k$. But $\dim A < \dim V_1 < \dim B \leq \dim F^3 = 3$. If $\dim A = 1$, then M_n is isomorphic to $\{A/A, V_1/A, \dots, V_n/A, B/A\}$ which is a sublattice of $\operatorname{Sub}(B/A)$, with $\dim B/A = 2$. If $\dim A = 0$, then:

$$\dim B = \dim(V_1 \oplus V_2) = \dim V_1 + \dim V_2 = 2 \cdot \dim V_1.$$

Thus $\dim B$ is even, moreover $\dim B \leq 3$, hence $\dim B = 2$.

In both cases M_n is a sublattice of $\operatorname{Sub} E$ for some F-vector space E of dimension two. But $\operatorname{Sub} E \cong M_{1+\operatorname{card} F}$, thus $n \leq 1 + \operatorname{card} F$. $\square$

Corollary 4.20. *Let F be a finite field and let $n > 1 + \operatorname{card} F$. Then:*

$$\operatorname{crit}(\mathcal{M}_n; \mathbf{Var}(\operatorname{Sub} F^3)) = \aleph_2;$$
$$\operatorname{crit}(\mathcal{M}_n; \mathbf{Var}_0(\operatorname{Sub} F^3)) = \aleph_2;$$
$$\operatorname{crit}(\mathcal{M}_n^{0,1}; \mathbf{Var}(\operatorname{Sub} F^3)) = \aleph_2;$$
$$\operatorname{crit}(\mathcal{M}_n^{0,1}; \mathbf{Var}_{0,1}(\operatorname{Sub} F^3)) = \aleph_2.$$

Proof. By Lemma 4.19, $M_n \notin \mathbf{Var}(\operatorname{Sub} F^3)$, moreover simple lattices of $\mathbf{Var}(\operatorname{Sub} F^3)$ are of length at most three. Thus, by Lemma 4.15:

$$\operatorname{crit}(\mathcal{M}_n^{0,1}; \mathbf{Var}(\operatorname{Sub} F^3)) \leq \aleph_2. \tag{4.4}$$

As each simple lattice of $\mathcal{M}_n$ is of length at most two, it follows from Theorem 3.11 that

$$\operatorname{crit}(\mathcal{M}_n; \mathbf{Var}_0(\operatorname{Sub} F^n)) \geq \aleph_2, \quad \text{and} \operatorname{crit}(\mathcal{M}_n^{0,1}; \mathbf{Var}_{0,1}(\operatorname{Sub} F^n)) \geq \aleph_2. \tag{4.5}$$

All the other desired equalities are consequences of (4.4), (4.5). $\square$

Corollary 4.21. *Let F and K be finite fields. If $\operatorname{card} F > \operatorname{card} K$ then:*

$$\operatorname{crit}(\mathbf{Var}(\operatorname{Sub} F^3); \mathbf{Var}(\operatorname{Sub} K^3)) = \aleph_2;$$
$$\operatorname{crit}(\mathbf{Var}(\operatorname{Sub} F^3); \mathbf{Var}_0(\operatorname{Sub} K^3)) = \aleph_2;$$
$$\operatorname{crit}(\mathbf{Var}_{0,1}(\operatorname{Sub} F^3); \mathbf{Var}(\operatorname{Sub} K^3)) = \aleph_2;$$
$$\operatorname{crit}(\mathbf{Var}_{0,1}(\operatorname{Sub} F^3); \mathbf{Var}_{0,1}(\operatorname{Sub} K^3)) = \aleph_2.$$

Proof. By Lemma 4.19, $M_{1+\operatorname{card} F} \notin \mathbf{Var}(\operatorname{Sub} K^3)$, moreover simple lattices of $\mathbf{Var}(\operatorname{Sub} K^3)$ are of length at most three. Thus, by Lemma 4.15:

$$\operatorname{crit}(\mathbf{Var}_{0,1}(\operatorname{Sub} F^3); \mathbf{Var}(\operatorname{Sub} K^3)) \leq \aleph_2. \tag{4.6}$$

As each simple lattice of $\mathbf{Var}(\operatorname{Sub} F^3)$ is of length at most three, it follows from Theorem 3.11 that:

$$\operatorname{crit}(\mathbf{Var}(\operatorname{Sub} F^3); \mathbf{Var}_0(\operatorname{Sub} K^n)) \geq \aleph_2, \tag{4.7}$$
$$\operatorname{crit}(\mathbf{Var}_{0,1}(\operatorname{Sub} F^3); \mathbf{Var}_{0,1}(\operatorname{Sub} K^n)) \geq \aleph_2. \tag{4.8}$$

All the other desired equalities are consequences of (4.6), (4.7), (4.8). $\square$

Lemma 4.22. *Let $\mathcal{V}$ be a finitely generated variety of lattices (resp., a finitely generated variety of lattices with 0), let $m \geq 2$ an integer. Assume that for each simple lattice K of $\mathcal{V}$, there do not exist $b_0, b_1, \dots, b_{m-1} > u$ in K such that $b_i \wedge b_j = u$ (resp., $b_0, b_1, \dots, b_{m-1} > 0$ such that $b_i \wedge b_j = 0$), for all $0 \leq i < j \leq m-1$. Then $\operatorname{crit}(\mathcal{M}_{2m-1}^{0,1}; \mathcal{V}) \leq \aleph_2$.*

Proof. Set $n = 2m - 1 \geq 3$. Let $\vec{A} = (A_P, f_{P,Q})_{P \subseteq Q \text{ in } I_n}$ be the direct system of $\mathcal{M}_n^{0,1}$ introduced just before Lemma 4.5. Assume that $\operatorname{crit}(\mathcal{M}_n^{0,1}; \mathcal{V}) > \aleph_2$. By Lemma 4.12, there exists a congruence-lifting $\vec{B} = (B_P, g_{P,Q})_{P \subseteq Q \text{ in } I_n}$ of $\operatorname{Con_c} \circ \vec{A}$ in $\mathcal{V}$. Let $\vec{\xi} = (\xi_P)_{P \in I_n} \colon \operatorname{Con_c} \circ \vec{A} \to \operatorname{Con_c} \circ \vec{B}$ be a natural equivalence. Taking a sublattice of $B_\emptyset$, we can assume that $B_\emptyset$ is a chain $u < v$. Moreover, as the map $f_{P,Q}$ is an inclusion map, we can assume that $g_{P,Q}$ is an inclusion map, for all $P \subseteq Q$ in I_n.

Let $x \in \underline{n}$. By Lemma 4.5, $\Theta_{B_{\{x\}}}(u,v)$ is the largest congruence of $B_{\{x\}}$. Thus:

$$\Theta_{B_{\{x\}}}(u,v) = \xi_{\{x\}}(\Theta_{A_{\{x\}}}(0,a_x)) \vee \xi_{\{x\}}(\Theta_{A_{\{x\}}}(a_x,1))$$

Therefore there exist $t_0^x = u < t_1^x < \cdots < t_{r+1}^x = v$ in $B_{\{x\}}$ such that, for all $0 \le i \le r$:

$$\text{either } (t_i^x, t_{i+1}^x) \in \xi_{\{x\}}(\Theta_{A_{\{x\}}}(0,a_x)) \text{ or } (t_i^x, t_{i+1}^x) \in \xi_{\{x\}}(\Theta_{A_{\{x\}}}(a_x,1))$$

Set $b_x = t_1^x$. Put:

$$X' = \{x \in \underline{n} \mid \Theta_{B_{\{x\}}}(u,b_x) = \xi_{\{x\}}(\Theta_{A_{\{x\}}}(0,a_x))\}$$
$$X'' = \{x \in \underline{n} \mid \Theta_{B_{\{x\}}}(u,b_x) = \xi_{\{x\}}(\Theta_{A_{\{x\}}}(a_x,1))\}.$$

By symmetry we can assume that $\operatorname{card} X' \ge \operatorname{card} X''$ (we can replace the diagram $\vec{A}$ by its dual if required). As $\underline{n} = X' \cup X''$ and $\operatorname{card} \underline{n} = n = 2m-1$, $\operatorname{card} X' \ge m$. Let $x,y \in X'$ distinct, it follows from Lemma 4.5(2) that $b_x \wedge b_y = u$. So we obtain a family of elements $(b_x)_{x \in X'}$ greater than u such that $b_x \wedge b_y = u$ (resp., $b_x \wedge b_y = u = 0$) for all $x \neq y$ in X', a contradiction. $\square$

With a similar proof using Lemma 4.13 instead of Lemma 4.12 we obtain the following lemma.

Lemma 4.23. *Let $\mathcal{V}$ be a variety of lattices (resp., a variety of lattices with 0), let $m \ge 2$ an integer. Assume that for each simple lattice K of $\mathcal{V}$, there do not exist $b_0, b_1, \dots, b_{m-1} > u$ in K such that $b_i \wedge b_j = u$ (resp., $b_0, b_1, \dots, b_{m-1} > 0$ such that $b_i \wedge b_j = 0$), for all $0 \le i < j \le m-1$. Then $\operatorname{crit}(\mathcal{M}_{2m-1}^{0,1}; \mathcal{V}) \le \aleph_3$.*

Theorem 4.24. *Let $\mathcal{V}$ be either a finitely generated variety of lattices or a finitely generated variety of lattices with 0. If $M_3 \in \mathcal{V}$ then:*

$$\operatorname{crit}(\mathcal{M}_\omega; \mathcal{V}) = \aleph_2;$$
$$\operatorname{crit}(\mathcal{M}_\omega^0; \mathcal{V}) = \aleph_2.$$

Let $\mathcal{V}$ be a finitely generated variety of bounded lattices. If $M_3 \in \mathcal{V}$ then:

$$\operatorname{crit}(\mathcal{M}_\omega^{0,1}; \mathcal{V}) = \aleph_2.$$

Proof. Let $\mathcal{V}$ be a finitely generated variety of lattices, let m be the maximal cardinality of a simple lattice of $\mathcal{V}$. Thus the assumptions of Lemma 4.22 are satisfied, so *a fortiori* $\operatorname{crit}(\mathcal{M}_{2m-1}^{0,1}; \mathcal{V}) \le \aleph_2$, and so $\operatorname{crit}(\mathcal{M}_\omega^{0,1}; \mathcal{V}) \le \aleph_2$.

Denote by $\mathbb{F}_2$ the two-element field. Let $\mathcal{V}$ be a variety of lattices with 0 (resp., with 0 and 1), such that $M_3 \in \mathcal{V}$. The variety $\mathcal{M}_\omega$ is locally finite, thus all finitely generated lattices of $\mathcal{M}_\omega$ are of finite length. Moreover all simple lattices of $\mathcal{M}_\omega$ have length at most two. Thus, by Theorem 3.11:

$$\operatorname{crit}(\mathcal{M}_\omega; \mathbf{Var}_0(\operatorname{Sub} \mathbb{F}_2^2)) \ge \aleph_2 \text{ (resp., } \operatorname{crit}(\mathcal{M}_\omega^{0,1}; \mathbf{Var}_{0,1}(\operatorname{Sub} \mathbb{F}_2^2)) \ge \aleph_2).$$

Moreover $\operatorname{Sub} \mathbb{F}_2^2 \cong M_3$, so $\operatorname{crit}(\mathcal{M}_\omega; \mathcal{V}) \ge \aleph_2$. $\square$

5. Acknowledgment

I thank Friedrich Wehrung whose help, advise, and careful reading of the paper, led to several improvements both in form and substance. I also thank Miroslav Ploščica for the stimulating discussions that led to the diagram $\vec{A}$ introduced in Section 4. Partially supported by the institutional grant MSM 0021620839.

References

[1] S. Z. Ditor, *Cardinality questions concerning semilattices of finite breadth*, Discrete Math. **48**, no. 1 (1984), 47–59.

[2] B. A. Davey and B. Sands, *An application of Whitman's condition to lattices with no infinite chains*, Algebra Universalis **7**, no. 2 (1977), 171–178.

[3] P. Erdös, A. Hajnal, A. Máté and R. Rado, "Combinatorial Set Theory: Partition Relations for Cardinals". Studies in Logic and the Foundations of Mathematics, 106. North-Holland Publishing Co., Amsterdam, 1984. 347 p.

[4] K. D. Fryer and I. Halperin, *The von Neumann coordinatization theorem for complemented modular lattices*, Acta Sci. Math. (Szeged) **17** (1956), 203–249.

[5] P. Gillibert, *Critical points of pairs of varieties of algebras*, Internat. J. Algebra Comput. **19**, no. 1 (2009), 1–40.

[6] G. Grätzer, "General Lattice Theory. Second edition", new appendices by the author with B. A. Davey, R. Freese, B. Ganter, M. Greferath, P. Jipsen, H. A. Priestley, H. Rose, E. T. Schmidt, S. E. Schmidt, F. Wehrung, and R. Wille. Birkhäuser Verlag, Basel, 1998. xx+663 p.

[7] K. R. Goodearl, "Von Neumann Regular Rings. Second edition". Robert E. Krieger Publishing Co., Inc., Malabar, FL, 1991. xviii+412 p.

[8] A. Hajnal and A. Máté, *Set mappings, partitions, and chromatic numbers*, Logic Colloquium '73 (Bristol, 1973), p. 347–379. Studies in Logic and the Foundations of Mathematics, Vol. **80**, North-Holland, Amsterdam, 1975.

[9] F. Micol, "On representability of ∗-regular rings and modular ortholattices". PhD thesis, TU Darmstadt, January 2003. Available online at
`http://elib.tu-darmstadt.de/diss/000303/diss.pdf` .

[10] M. Ploščica, *Separation properties in congruence lattices of lattices*, Colloq. Math., **83** (2000), 71–84.

[11] M. Ploščica, *Dual spaces of some congruence lattices*, Topology and its Applications, **131** (2003), 1–14.

[12] M. Ploščica, J. Tůma, and F. Wehrung, *Congruence lattices of free lattices in nondistributive varieties*, Colloq. Math. **76**, no. 2 (1998), 269–278.

[13] F. Wehrung, *The dimension monoid of a lattice*, Algebra Universalis **40**, no. 3 (1998), 247–411.

[14] F. Wehrung, *A uniform refinement property for congruence lattices*, Proc. Amer. Math. Soc. **127**, no. 2 (1999), 363–370.

[15] F. Wehrung, *A solution to Dilworth's congruence lattice problem*, Adv. Math. **216**, no. 2 (2007), 610–625.

LMNO, CNRS UMR 6139, Département de Mathématiques, BP 5186, Université de Caen, Campus 2, 14032 Caen cedex, France

Charles University in Prague, Faculty of Mathematics and Physics, Department of Algebra, Sokolovska 83, 186 00 Prague, Czech Republic.

E-mail address: `pierre.gillibert@math.unicaen.fr, pgillibert@yahoo.fr`

URL: `http://www.math.unicaen.fr/~giliberp/`