

AN INFINITE COMBINATORIAL STATEMENT WITH A POSET PARAMETER

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ABSTRACT. We introduce an extension, indexed by a partially ordered set P and cardinal numbers κ, λ , denoted by $(\kappa, <\lambda) \rightsquigarrow P$, of the classical relation $(\kappa, n, \lambda) \rightarrow \rho$ in infinite combinatorics. By definition, $(\kappa, n, \lambda) \rightarrow \rho$ holds if every map $F: [\kappa]^n \rightarrow [\kappa]^{<\lambda}$ has a ρ -element free set. For example, Kuratowski's Free Set Theorem states that $(\kappa, n, \lambda) \rightarrow n+1$ holds iff $\kappa \geq \lambda^{+n}$, where λ^{+n} denotes the n -th cardinal successor of an infinite cardinal λ . By using the $(\kappa, <\lambda) \rightsquigarrow P$ framework, we present a self-contained proof of the first author's result that $(\lambda^{+n}, n, \lambda) \rightarrow n+2$, for each infinite cardinal λ and each positive integer n , which solves a problem stated in the 1985 monograph of Erdős, Hajnal, Máté, and Rado. Furthermore, by using an order-dimension estimate established in 1971 by Hajnal and Spencer, we prove the relation $(\lambda^{+(n-1)}, r, \lambda) \rightarrow 2 \lfloor \frac{1}{2}(1-2^{-r})^{-n/r} \rfloor$, for every infinite cardinal λ and all positive integers n and r with $2 \leq r < n$. For example, $(\aleph_{210}, 4, \aleph_0) \rightarrow 32,768$. Other order-dimension estimates yield relations such as $(\aleph_{109}, 4, \aleph_0) \rightarrow 257$ (using an estimate by Füredi and Kahn) and $(\aleph_7, 4, \aleph_0) \rightarrow 10$ (using an exact estimate by Dushnik).

1. INTRODUCTION

The present paper deals with finding large free sets for infinite set mappings of finite order. For a set X and a cardinal λ , we denote by $[X]^\lambda$ (resp., $[X]^{<\lambda}$) the set of all subsets of X with λ elements (resp., less than λ elements). For cardinals κ and λ , a subset \mathcal{D} of the powerset $\mathfrak{P}(\kappa)$ of κ , and a map $F: \mathcal{D} \rightarrow [\kappa]^{<\lambda}$, we say that a subset H of κ is *free with respect to F* if $F(X) \cap H \subseteq X$ for each $X \in \mathfrak{P}(H) \cap \mathcal{D}$. For cardinals κ, λ, ρ , and for a positive integer r , the statement $(\kappa, r, \lambda) \rightarrow \rho$ holds if every map $F: [\kappa]^r \rightarrow [\kappa]^{<\lambda}$ (we say a *set mapping of order r*) has a ρ -element free set. Likewise, $(\kappa, <\omega, \lambda) \rightarrow \rho$ means that every map $F: [\kappa]^{<\omega} \rightarrow [\kappa]^{<\lambda}$ has a ρ -element free set. Denote by λ^{+n} the n -th successor of an infinite cardinal λ . Kuratowski's Free Set Theorem (see [18] or [6, Theorem 46.1]) states that $(\kappa, n, \lambda) \rightarrow n+1$ iff $\kappa \geq \lambda^{+n}$, for all infinite cardinals κ and λ and every non-negative integer n .

Whether larger free subsets can be found leads to unexpected discoveries. An argument that originates in Lázár [19] shows that the relation $(\lambda^+, 1, \lambda) \rightarrow \lambda^+$ holds for each infinite cardinal λ (cf. [6, Corollary 44.2]). Therefore, the relation $(\lambda^+, 1, \lambda) \rightarrow m$ holds *a fortiori*, for each positive integer m . Further, Hajnal-Máté [12] and Hajnal [6], respectively, proved that the relation $(\lambda^{+r}, r, \lambda) \rightarrow m$

Date: April 23, 2010.

2000 Mathematics Subject Classification. 03E05, 06A07, 05A05, 05D10, 06A05.

Key words and phrases. Kuratowski's Free Set Theorem; Kuratowski index; free set; cardinal; poset; order-dimension; breadth; truncated cube.

This work was partially supported by the institutional grant MSM 0021620839.

holds for every infinite cardinal λ , every $r \in \{2, 3\}$, and every integer $m > r$ (cf. [6, Theorem 46.2]).

In the presence of the Generalized Continuum Hypothesis GCH, we can say more. Indeed, it follows from [6, Theorem 45.5] that the relation $(\lambda^{+r}, r, \lambda) \rightarrow \lambda^+$ holds for every integer $r \geq 2$ and every infinite cardinal λ .

On the other hand, without assuming GCH, the situation gains a touch of strangeness. Set $t_0 := 5$, $t_1 := 7$, and for each positive integer n , t_{n+1} is the least positive integer such that $t_{n+1} \rightarrow (t_n, 7)^5$ (the latter notation meaning that for each $f: [t_{n+1}]^5 \rightarrow \{0, 1\}$, either there exists a t_n -element subset X of t_{n+1} such that $f^{\llbracket X \rrbracket^5} = \{0\}$ or there exists a 7-element subset X of t_{n+1} such that $f^{\llbracket X \rrbracket^5} = \{1\}$). The existence of the sequence $(t_n \mid n < \omega)$ is ensured by Ramsey's Theorem. It is established in Komjáth and Shelah [17] that for each positive integer n , there exists a generic extension of the universe in which $(\aleph_n, 4, \aleph_0) \not\rightarrow t_n$. In particular, there exists a generic extension of the universe in which $(\aleph_4, 4, \aleph_0) \not\rightarrow t_4$.

By using algebraic tools of completely different nature and purpose introduced in [8, 9] called *compatible norm-coverings*, the first author established in [8, Théorème 3.3.13] the relation

$$(\lambda^{+r}, r, \lambda) \rightarrow r + 2 \quad (1.1)$$

for each positive integer r and each infinite cardinal λ , thus improving by one the cardinality of the free set given by Kuratowski's Free Set Theorem, and thus solving in the affirmative the question, raised in [6, page 285], whether $(\aleph_4, 4, \aleph_0) \rightarrow 6$. In the present paper, we develop a self-contained approach to such questions, and we extend the methods in order to find further large free sets results. In particular, we establish the relation

$$(\lambda^{+(n-1)}, r, \lambda) \rightarrow 2^{\lfloor \frac{1}{2}(1-2^{-r})^{-n/r} \rfloor}, \quad (1.2)$$

for each infinite cardinal λ and for all positive integers n and r with $2 \leq r < n$; here $\lfloor x \rfloor$ denotes the largest integer below any real number x .

Our main idea is to extend the $(\kappa, r, \lambda) \rightarrow \rho$ notation by introducing a partially ordered set (poset) parameter, thus defining the notation $(\kappa, < \lambda) \rightsquigarrow P$, for cardinals κ , λ and a poset P , see Definition 3.1. In particular, the statements $(\kappa, < \lambda) \rightsquigarrow ([\rho]^{< \omega}, \subseteq)$ and $(\kappa, < \omega, \lambda) \rightarrow \rho$ are equivalent, for all cardinals κ , λ , and ρ (cf. Proposition 3.4). Then we define the *Kuratowski index* of a finite poset P as the least non-negative integer n such that $(\kappa^{+(n-1)}, < \kappa) \rightsquigarrow P$ holds for each infinite cardinal κ (cf. Definition 4.1), with a minor adjustment for antichains. This definition is tailored in order to ensure that the finite Boolean lattice $\mathfrak{B}(n)$ of all subsets of n has Kuratowski index n , for each positive integer n . Smaller posets have smaller Kuratowski index (Proposition 4.4) while the Kuratowski index function is subadditive on finite direct products (cf. Proposition 4.6). As a corollary, the Kuratowski index is bounded above by the order-dimension (cf. Proposition 4.7).

We apply these results to the *truncated cubes* $B_m(\leq r)$ (cf. Notation 5.1). In particular, we present the first author's proof of the relation (1.1), which uses the fact that the order-dimension of $B_{n+2}(\leq n)$ (and thus also its Kuratowski index) is equal to $n + 1$. By using a further estimate of the order-dimension established in a 1971 paper by Spencer [20], we deduce the relation (1.2) (cf. Proposition 5.7). For example, $(\aleph_{210}, 4, \aleph_0) \rightarrow 32,768$.

Another estimate of the order-dimension of $B_m(\leq r)$, originating from a 1986 paper by Füredi and Kahn [7], not so good asymptotically but giving larger free

sets for smaller alephs (cf. Proposition 5.8), yields, for example, the relation

$$(\aleph_{109}, 4, \aleph_0) \rightarrow 257.$$

Finally, Dushnik's original exact estimate yields, for example, the relation

$$(\aleph_7, 4, \aleph_0) \rightarrow 10.$$

2. BASIC CONCEPTS

2.1. Set theory. We shall use basic set-theoretical notation and terminology about ordinals and cardinals. We denote by $f^{\llcorner}(X)$, or $f^{\llcorner}X$ (resp., $f^{-1}X$) the image (resp., inverse image) of a set X under f . Cardinals are initial ordinals. We denote by $\text{cf } \alpha$ the cofinality of an ordinal α . We denote by $\omega := \{0, 1, 2, \dots\}$ the first limit ordinal, mostly denoted by \aleph_0 in case it is viewed as a cardinal. We identify every non-negative integer n with the finite set $\{0, 1, \dots, n-1\}$ (so $0 = \emptyset$). We denote by κ^+ the successor cardinal of a cardinal κ , and we define κ^{+n} , for a non-negative integer n , by $\kappa^{+0} := \kappa$ and $\kappa^{+(n+1)} = (\kappa^{+n})^+$.

We denote by $\mathfrak{P}(X)$ the powerset of a set X , and we set

$$\begin{aligned} [X]^\kappa &:= \{Y \in \mathfrak{P}(X) \mid \text{card } Y = \kappa\}, \\ [X]^{<\kappa} &:= \{Y \in \mathfrak{P}(X) \mid \text{card } Y < \kappa\}, \\ [X]^{\leq \kappa} &:= \{Y \in \mathfrak{P}(X) \mid \text{card } Y \leq \kappa\}, \end{aligned}$$

for every cardinal κ .

2.2. Partially ordered sets (posets). All our posets will be nonempty. For posets P and Q , a map $f: P \rightarrow Q$ is *isotone* if $x \leq y$ implies that $f(x) \leq f(y)$, for all $x, y \in P$.

We denote by 0_P the least element of P if it exists. An element p in a poset P is *join-irreducible* if $p = \bigvee X$ implies that $p \in X$, for every (possibly empty) finite subset X of P ; we denote by $J(P)$ the set of all join-irreducible elements of P , endowed with the induced partial ordering. We set

$$\begin{aligned} Q \downarrow X &:= \{q \in Q \mid (\exists x \in X)(q \leq x)\}, \\ Q \Downarrow X &:= \{q \in Q \mid (\exists x \in X)(q < x)\}, \end{aligned}$$

for all subsets Q and X of P ; in case $X = \{a\}$ is a singleton, then we shall write $Q \downarrow a$ (resp., $Q \Downarrow a$) instead of $Q \downarrow \{a\}$ (resp., $Q \Downarrow \{a\}$). We set $J_P(a) := J(P) \downarrow a$, for each $a \in P$. A subset Q of P is a *lower subset of P* if $P \downarrow Q = Q$. We say that P is a *tree* if P has a smallest element and $P \downarrow a$ is a chain for each $a \in P$.

3. AN INFINITE COMBINATORIAL STATEMENT WITH A POSET PARAMETER

The statement referred to in the section title is the following.

Definition 3.1. For cardinals κ, λ and a poset P , let $(\kappa, <\lambda) \rightsquigarrow P$ hold if for every mapping $F: \mathfrak{P}(\kappa) \rightarrow [\kappa]^{<\lambda}$, there exists a one-to-one map $f: P \rightarrow \kappa$ such that

$$F(f^{\llcorner}(P \downarrow x)) \cap f^{\llcorner}(P \downarrow y) \subseteq f^{\llcorner}(P \downarrow x), \quad \text{for all } x \leq y \text{ in } P. \quad (3.1)$$

In most cases, we shall use the symbol $(\kappa, <\lambda) \rightsquigarrow P$ only in case P is *lower finite*, in which case it is sufficient to assume that the mapping F is defined only on $[\kappa]^{<\omega}$ (we can always set $F(X) := \emptyset$ for infinite X). Once this is set, it is of course sufficient to assume that F is *isotone* (replace $F(X)$ by $\bigcup(F(Y) \mid Y \subseteq X)$).

Lemma 3.2. *Let P and Q be posets, with Q lower finite, and let κ, λ be cardinals. If $(\kappa, <\lambda) \rightsquigarrow Q$ and P embeds into Q , then $(\kappa, <\lambda) \rightsquigarrow P$.*

Proof. We may assume that P is a sub-poset of Q . Let $F: [\kappa]^{<\omega} \rightarrow [\kappa]^{<\lambda}$ be isotone. By assumption, there exists a one-to-one map $g: Q \rightarrow \kappa$ such that

$$F(g^{``}(Q \downarrow x)) \cap g^{``}(Q \downarrow y) \subseteq g^{``}(Q \downarrow x), \quad \text{for all } x \leq y \text{ in } Q.$$

The restriction f of g to P is one-to-one. As F is isotone and $f^{``}(P \downarrow x) \subseteq g^{``}(Q \downarrow x)$ for each $x \in P$, (3.1) is obviously satisfied. \square

The following lemma states that the $(\kappa, <\lambda) \rightsquigarrow P$ relation can be “verified on the join-irreducible elements”.

Lemma 3.3. *Let P be a lower finite poset and let λ and κ be infinite cardinals. Then $(\kappa, <\lambda) \rightsquigarrow P$ iff for every isotone mapping $G: [\kappa]^{<\omega} \rightarrow [\kappa]^{<\lambda}$, there exists a one-to-one map $g: J(P) \rightarrow \kappa$ such that*

$$G(g^{``}J_P(x)) \cap g^{``}J_P(y) \subseteq g^{``}J_P(x), \quad \text{for all } x \leq y \text{ in } P. \quad (3.2)$$

Proof. It is trivial that $(\kappa, <\lambda) \rightsquigarrow P$ entails the given condition. For the converse, we shall fix a bijection $\varphi: [\kappa]^{<\omega} \rightarrow \kappa$, and set

$$\Phi(X) := \bigcup \varphi^{-1}X, \quad \text{for each } X \in \mathfrak{P}(\kappa).$$

That is, $\Phi(X)$ is the union of all finite subsets Y of κ such that $\varphi(Y) \in X$. In particular, $\Phi(X)$ belongs to $[\kappa]^{<\lambda}$ whenever X belongs to $[\kappa]^{<\lambda}$. Now let $F: [\kappa]^{<\omega} \rightarrow [\kappa]^{<\lambda}$ be an isotone map. We can define a map $G: [\kappa]^{<\omega} \rightarrow [\kappa]^{<\lambda}$ by the rule

$$G(X) := \Phi(F(\{\varphi(Y) \mid Y \subseteq X\})), \quad \text{for each } X \in [\kappa]^{<\omega}.$$

By assumption, there exists a one-to-one map $g: J(P) \rightarrow \kappa$ which satisfies the condition (3.2). As P is lower finite, we can define a map $f: P \rightarrow \kappa$ by the rule

$$f(x) := \varphi(g^{``}J_P(x)), \quad \text{for each } x \in P.$$

As P is lower finite, every element of P is the join of all elements of $J(P)$ below it. Hence, as both g and φ are one-to-one, f is one-to-one as well.

Now we prove that the condition (3.1) holds. Let $x \leq y$ in P and let $u \in P \downarrow x$ such that $f(u) \in F(f^{``}(P \downarrow x))$, we must prove that $u \leq x$. As

$$f^{``}(P \downarrow x) = \{\varphi(g^{``}J_P(v)) \mid v \in P \downarrow x\} \subseteq \{\varphi(Y) \mid Y \subseteq g^{``}J_P(x)\}$$

and F is isotone, we get

$$\varphi(g^{``}J_P(u)) = f(u) \in F(f^{``}(P \downarrow x)) \subseteq F(\{\varphi(Y) \mid Y \subseteq g^{``}J_P(x)\}),$$

thus $g^{``}J_P(u) \subseteq \Phi(F(\{\varphi(Y) \mid Y \subseteq g^{``}J_P(x)\})) = G(g^{``}J_P(x))$. This means that for each $p \in J_P(u)$, $g(p)$ belongs to $G(g^{``}J_P(x))$. As $g(p)$ also belongs to $g^{``}J_P(y)$ (because $p \leq u \leq y$), it follows, using (3.2), that $g(p) \in g^{``}J_P(x)$, and so $p \leq x$. As this holds for each $p \in J_P(u)$, we obtain that $u \leq x$, as was to be proved. \square

We shall now recall a few known facts about the relation $(\kappa, <\omega, \lambda) \rightarrow \rho$ (cf. Section 1). In case $\lambda \geq \aleph_1$ and $\rho \geq \aleph_0$, the existence of κ such that $(\kappa, <\omega, \lambda) \rightarrow \rho$ is a large cardinal axiom, that entails the existence of $0^\#$ in case $\rho \geq \aleph_1$ (cf. [3], and also [15, 16] for further related consistency strength results). The relation $(\kappa, <\omega, \lambda) \rightarrow \rho$ follows from the infinite partition property $\kappa \rightarrow (\theta)_2^{<\omega}$ (existence of the θ^{th} Erdős cardinal) where $\theta := \max\{\rho, \lambda^+\}$, see [6, Theorem 45.2] and the

discussion preceding it. This notation is related to ours *via* the following easy result.

Proposition 3.4. *The statements $(\kappa, <\lambda) \rightsquigarrow ([\rho]^{<\omega}, \subseteq)$ and $(\kappa, <\omega, \lambda) \rightarrow \rho$ are equivalent, for every cardinal ρ and all infinite cardinals κ, λ .*

Proof. We use the characterization of the \rightsquigarrow relation given by Lemma 3.3. First observe that $J([\rho]^{<\omega}) = \{\{\xi\} \mid \xi < \rho\}$. Assume that the statement $(\kappa, <\lambda) \rightsquigarrow [\rho]^{<\omega}$ holds, and let $F: [\kappa]^{<\omega} \rightarrow [\kappa]^{<\lambda}$. By using the easy direction of Lemma 3.3, we obtain a one-to-one map $f: \rho \rightarrow \kappa$ such that, putting $H := f^{<}\!(\rho)$,

$$F(f^{<}\!(p)) \cap H \subseteq f^{<}\!(p), \quad \text{for each } p \in [\rho]^{<\omega}. \quad (3.3)$$

This means that H is free with respect to F .

Conversely, assume that $(\kappa, <\omega, \lambda) \rightarrow \rho$ holds, and let $F: \mathfrak{P}(\kappa) \rightarrow [\kappa]^{<\lambda}$. By assumption, there exists $H \in [\kappa]^\rho$ which is free with respect to F . Then (3.3) holds, for any one-to-one map $f: \rho \rightarrow H$. Hence $(\kappa, <\lambda) \rightsquigarrow ([\rho]^{<\omega}, \subseteq)$ holds. \square

The following observation illustrates the difficulties of checking Definition 3.1 on arbitrary posets.

Proposition 3.5. *There exists no cardinal κ satisfying $(\kappa, <\aleph_1) \rightsquigarrow (\mathfrak{P}(\omega), \subseteq)$.*

Proof. Observe that $J(\mathfrak{P}(\omega)) = \{\{n\} \mid n < \omega\}$. Let $X \equiv_{\text{fin}} Y$ hold if the symmetric difference $X \Delta Y$ is finite, for all subsets X and Y of κ . Let Δ be a set that meets the \equiv_{fin} -equivalence class $[X]_{\equiv_{\text{fin}}}$ of X in exactly one point, for each $X \in \mathfrak{P}(\kappa)$, and denote by $F(X)$ the unique element of $[X]_{\equiv_{\text{fin}}} \cap \Delta$ if X is at most countable, \emptyset otherwise. If the relation $(\kappa, <\aleph_1) \rightsquigarrow (\mathfrak{P}(\omega), \subseteq)$ holds, then there is a one-to-one map $f: \omega \rightarrow \kappa$ such that $F(f^{<}\!(p)) \cap f^{<}\!(q) \subseteq f^{<}\!(p)$ for all $p \subseteq q$ in $\mathfrak{P}(\omega)$. As $F(f^{<}\!(\omega)) \equiv_{\text{fin}} f^{<}\!(\omega)$, there exists $m < \omega$ such that $f^{<}\!(\omega \setminus m) \subseteq F(f^{<}\!(\omega))$. Put $p := \omega \setminus (m + 1)$. It follows from the relation $f^{<}\!(p) \equiv_{\text{fin}} f^{<}\!(\omega)$ that $F(f^{<}\!(p)) = F(f^{<}\!(\omega))$, thus

$$f(m) \in F(f^{<}\!(\omega)) \cap f^{<}\!(\omega) = F(f^{<}\!(p)) \cap f^{<}\!(\omega) \subseteq f^{<}\!(p),$$

while $f(m) \notin f^{<}\!(p)$, a contradiction. \square

4. THE KURATOWSKI INDEX OF A FINITE POSET

Definition 4.1. A non-negative integer n is a *Kuratowski index* of a finite poset P if either P is an antichain, or P is not an antichain, $n > 0$, and the relation $(\kappa^{+(n-1)}, <\kappa) \rightsquigarrow P$ holds for each infinite cardinal κ . Furthermore, we shall call *the Kuratowski index of P* the least such n if it exists (we shall see shortly that it does), and we shall denote it by $\text{kur}(P)$.

The reason of our adjustment of Definition 4.1 for antichains lies essentially in the fact that the relation $(\kappa, <\lambda) \rightsquigarrow P$ trivially holds for every antichain P and all cardinals κ, λ such that $\kappa \geq \text{card } P$.

Proposition 4.2. *The number of join-irreducible elements in P is a Kuratowski index of P , for each finite poset P . Hence, $\text{kur}(P)$ is defined and $\text{kur}(P) \leq \text{card } J(P)$.*

Proof. The statement is trivial in case P is an antichain, so suppose that P is not an antichain. Set $n := \text{card } J(P)$, let λ be an infinite cardinal, and set $\kappa := \lambda^{+(n-1)}$. It follows from Kuratowski's Free Set Theorem [18] that the relation $(\kappa, <\lambda) \rightsquigarrow (\mathfrak{P}(n), \subseteq)$ holds (cf. Proposition 3.4). As the assignment $x \mapsto J_P(x)$ defines an

embedding from P into $(\mathfrak{P}(J(P)), \subseteq)$, it follows from Lemma 3.2 that the relation $(\kappa, <\lambda) \rightsquigarrow P$ holds. \square

Proposition 4.3. *Let T be a well-founded tree and let κ be an infinite cardinal such that $\text{card} T \leq \kappa$ and $\text{card}(T \downarrow p) < \text{cf } \kappa$ for each $p \in T$. Then the relation $(\kappa, <\kappa) \rightsquigarrow T$ holds.*

Proof. As T is well-founded, there exists a unique map ρ from T to the ordinals (the rank map) such that $\rho(q) = \{\rho(p) \mid p \in T \Downarrow q\}$ for each $q \in T$. The range θ of the map ρ is an ordinal and $\text{card } \theta \leq \kappa$. Fix a partition $(I_\eta \mid \eta < \theta)$ of κ such that $\text{card } I_\eta = \kappa$ for each $\eta < \theta$. Fix also a one-to-one map $\nu: T \rightarrow \kappa$. We define a strict ordering \triangleleft on T (lexicographical product) by

$$p \triangleleft q \iff (\text{either } \rho(p) < \rho(q) \text{ or } (\rho(p) = \rho(q) \text{ and } \nu(p) < \nu(q))).$$

Observe that \triangleleft is a strict well-ordering of T and that it extends the original strict ordering $<$. Let $F: \mathfrak{P}(\kappa) \rightarrow [\kappa]^{<\kappa}$. We define, by \triangleleft -induction, a map $f: T \rightarrow \kappa$, as follows. Let $q \in T$ and suppose that $f(p)$ has been defined and belongs to $I_{\rho(p)}$, for each $p \in T$ such that $p \triangleleft q$. It follows from the assumptions on T that the set

$$X_q := \{f(p) \mid p \in T, \rho(p) = \rho(q), \nu(p) < \nu(q)\} \cup \bigcup (F(f^{\triangleleft}(T \downarrow p)) \mid p \in T \Downarrow q)$$

(where $T \Downarrow q$ is evaluated with respect to the original ordering of T) has cardinality less than κ . Hence, $I_{\rho(q)} \setminus X_q$ is nonempty. We pick $f(q) \in I_{\rho(q)} \setminus X_q$.

We claim that f is one-to-one. Indeed let $p, q \in T$ distinct, we prove that $f(p) \neq f(q)$. We may assume that $p \triangleleft q$. If $\rho(p) = \rho(q)$, then $\nu(p) < \nu(q)$, so $f(p) \in X_q$ and the conclusion follows. If $\rho(p) < \rho(q)$, then the conclusion follows from $f(p) \in I_{\rho(p)}$, $f(q) \in I_{\rho(q)}$, and $I_{\rho(p)} \cap I_{\rho(q)} = \emptyset$.

Furthermore, for all $p < q$ in T , $f(q)$ does not belong to X_q , thus, *a fortiori*, not to $F(f^{\triangleleft}(T \downarrow p))$. Therefore, $F(f^{\triangleleft}(T \downarrow p)) \cap f^{\triangleleft}(T \downarrow q) \subseteq f^{\triangleleft}(T \downarrow p)$ for all $p \leq q$ in T . \square

From Lemma 3.2 we deduce immediately the following.

Proposition 4.4. *If a poset P embeds into a finite poset Q , then $\text{kur}(P) \leq \text{kur}(Q)$.*

The proof of the following result is inspired by the proof of Kuratowski's Free Set Theorem, see [6, Theorem 46.1]. It is closely related to [8, Lemme 3.3.7].

Lemma 4.5 (Product Lemma). *Let P and Q be lower finite posets with zero and let α, β, γ be infinite cardinals such that $\alpha \leq \beta \leq \gamma$ and $\text{card } Q < \text{cf } \alpha$. If $(\beta, <\alpha) \rightsquigarrow P$ and $(\gamma, <\beta^+) \rightsquigarrow Q$, then $(\gamma, <\alpha) \rightsquigarrow P \times Q$.*

Proof. We use the equivalent form of the \rightsquigarrow relation provided by Lemma 3.3. Let $F: \mathfrak{P}(\gamma) \rightarrow [\gamma]^{<\alpha}$. As $\beta \leq \gamma$, there exists a partition (U, V) of the set γ such that $\text{card } U = \beta$ and $\text{card } V = \gamma$. As $\alpha \leq \beta$, we can define a map $H: \mathfrak{P}(V) \rightarrow [V]^{<\beta^+}$ by the rule

$$H(Y) := V \cap \bigcup (F(X \cup Y) \mid X \in [U]^{<\omega}), \quad \text{for each } Y \in \mathfrak{P}(V).$$

As $(\gamma, <\beta^+) \rightsquigarrow Q$, there exists a one-to-one map $h: J(Q) \rightarrow V$ such that

$$(\forall y \leq y' \text{ in } Q)(H(h^{\triangleleft} J_Q(y)) \cap h^{\triangleleft} J_Q(y') \subseteq h^{\triangleleft} J_Q(y)). \quad (4.1)$$

As $\text{card } Q < \text{cf } \alpha$, we can define a map $G: \mathfrak{P}(U) \rightarrow [U]^{<\alpha}$ by the rule

$$G(X) := U \cap \bigcup (F(X \cup h^{\triangleleft} J_Q(y)) \mid y \in Q), \quad \text{for each } X \in \mathfrak{P}(U).$$

As $(\beta, <\alpha) \rightsquigarrow P$, there exists a one-to-one map $g: J(P) \rightarrow U$ such that

$$(\forall x \leq x' \text{ in } P)(G(g^{\llcorner} J_P(x)) \cap g^{\llcorner} J_P(x') \subseteq g^{\llcorner} J_P(x)). \quad (4.2)$$

As $J(P \times Q) = (J(P) \times \{0_Q\}) \cup (\{0_P\} \times J(Q))$ and γ is the disjoint union of U and V , we can define a one-to-one map $f: J(P \times Q) \rightarrow \gamma$ by the rule

$$\begin{cases} f(p, 0_Q) & := g(p), & \text{for each } p \in J(P); \\ f(0_P, q) & := h(q), & \text{for each } q \in J(Q). \end{cases}$$

Let $(x, y) \leq (x', y')$ in $P \times Q$, we verify that the following statement holds:

$$F(f^{\llcorner} J_{P \times Q}(x, y)) \cap f^{\llcorner} J_{P \times Q}(x', y') \subseteq f^{\llcorner} J_{P \times Q}(x, y). \quad (4.3)$$

So let $(p, q) \in J_{P \times Q}(x', y')$ such that $f(p, q) \in F(f^{\llcorner} J_{P \times Q}(x, y))$, that is, $f(p, q) \in F(g^{\llcorner} J_P(x) \cup h^{\llcorner} J_Q(y))$. We must prove that $(p, q) \leq (x, y)$. We separate cases.

Case 1. $q = 0_Q$. As

$$g(p) = f(p, q) \in F(g^{\llcorner} J_P(x) \cup h^{\llcorner} J_Q(y)) \cap U \subseteq G(g^{\llcorner} J_P(x))$$

while $g(p) \in g^{\llcorner} J_P(x')$, it follows from (4.2) that $g(p) \in g^{\llcorner} J_P(x)$, that is, $p \leq x$.

Case 2. $p = 0_P$. As

$$h(q) = f(p, q) \in F(g^{\llcorner} J_P(x) \cup h^{\llcorner} J_Q(y)) \cap V \subseteq H(h^{\llcorner} J_Q(y))$$

while $h(q) \in h^{\llcorner} J_Q(y')$, it follows from (4.1) that $h(q) \in h^{\llcorner} J_Q(y)$, that is, $q \leq y$. This completes the proof of (4.3). \square

Proposition 4.6. *The inequality $\text{kur}(P \times Q) \leq \text{kur}(P) + \text{kur}(Q)$ holds, for any finite posets P and Q with zero.*

Proof. We may assume that P and Q are both nonzero. Set $m := \text{kur}(P)$ and $n := \text{kur}(Q)$. Let κ be an infinite cardinal and set $\lambda := \kappa^{+m}$. As $(\kappa^{+(m-1)}, <\kappa) \rightsquigarrow P$ and $(\lambda^{+(n-1)}, <\lambda) \rightsquigarrow Q$, it follows from the Product Lemma (Lemma 4.5) that $(\lambda^{+(n-1)}, <\kappa) \rightsquigarrow P \times Q$. Now observe that $\lambda^{+(n-1)} = \kappa^{+(m+n-1)}$. \square

Denote by $\text{dim}(P)$ the *order-dimension* of a poset P , that is, the smallest number κ of chains such that P embeds into a product of κ chains. The order-dimension of a finite poset is finite. It follows immediately from Proposition 4.3 that the Kuratowski index of a nontrivial finite tree is 1. Hence, by applying Proposition 4.6, we obtain immediately the following upper bound for $\text{kur}(P)$.

Proposition 4.7. *Let n be a positive integer and let P be a finite poset. If P embeds, as a poset, into a product of n trees, then $\text{kur}(P) \leq n$. In particular, $\text{kur}(P) \leq \text{dim}(P)$.*

As $\text{dim}(P) \leq \text{card } J(P)$ (for the assignment $x \mapsto J_P(x)$ defines a meet-embedding from P into $\mathfrak{P}(J(P))$), this bound is sharper than the previously observed bound $\text{card } J(P)$ (cf. Proposition 4.2). In fact, it dates back to Baker [1] that $\text{dim}(P) \leq \text{wd } J(P)$, where wd denotes the *width* function (the width of a poset is the supremum of the cardinalities of all antichains of that poset). Hence,

$$\text{kur}(P) \leq \text{dim}(P) \leq \text{wd } J(P) \leq \text{card } J(P). \quad (4.4)$$

While Proposition 4.7 provides an upper bound for the Kuratowski index of a finite poset, our next result, Proposition 4.8, will provide a lower bound. The classical definition of breadth, see Baker [1] or Ditor [4, Section 4], runs as follows. Let n be a positive integer. A join-semilattice P has *breadth at most n* if for every nonempty

finite subset X of P , there exists a nonempty $Y \subseteq X$ with at most n elements such that $\bigvee X = \bigvee Y$. This is a particular case of the following definition of breadth, valid for every poset, and, in addition, *self-dual*: we say that a poset P has breadth at most n if for all x_i, y_i ($0 \leq i \leq n$) in P , if $x_i \leq y_j$ for all $i \neq j$ in $\{0, 1, \dots, n\}$, then there exists $i \in \{0, 1, \dots, n\}$ such that $x_i \leq y_i$. We denote by $\text{br}(P)$ the least non-negative integer n , if it exists, such that P has breadth at most n , and we call it *the breadth of P* .

Proposition 4.8. *Let P be a finite poset with zero such that $P \downarrow a$ is a join-semilattice for each $a \in P$. Then the inequality $\max\{\text{br}(P \downarrow a) \mid a \in P\} \leq \text{kur}(P)$ holds.*

Proof. We may assume that $P \neq \{0\}$. Hence $n := \max\{\text{br}(P \downarrow a) \mid a \in P\}$ is a positive integer. Pick $a \in P$ such that $n = \text{br}(P \downarrow a)$. There are $p_0, \dots, p_{n-1} \in P \downarrow a$ such that $p_i \not\leq \bigvee (p_j \mid j \neq i)$ for each $i < n$, where the join $\bigvee (p_j \mid j \neq i)$ is evaluated in $P \downarrow a$. The assignment $X \mapsto \bigvee (p_i \mid i \in X)$ defines a join-embedding from $\mathfrak{P}(n)$ into P , thus, by Proposition 4.4, $\text{kur}(P) \geq \text{kur}(\mathfrak{P}(n)) = n$ (the latter equality is part of the content of Kuratowski's Free Set Theorem, cf. Proposition 3.4). \square

As an immediate application of Propositions 4.7 and 4.8, we obtain the following.

Corollary 4.9. *Let n be a positive integer and let P be a product of n nontrivial finite trees. Then $\text{kur}(P) = n$.*

5. KURATOWSKI INDEXES OF TRUNCATED CUBES

In this section, we shall give estimates of Kuratowski indexes of *truncated cubes*, with applications to finding large free sets for set mappings of order greater than one.

Notation 5.1. For integers $r, k, r_0 \leq \dots \leq r_{k-1}$, and m such that $1 \leq r \leq m$ and $1 \leq r_j \leq m$ for each $j < k$, we define the *truncated m -dimensional cubes*

$$\begin{aligned} B_m(\leq r) &:= \{X \in \mathfrak{P}(m) \mid \text{either } \text{card } X \leq r \text{ or } X = m\}, \\ B_m(r_0, \dots, r_{k-1}) &:= \{X \in \mathfrak{P}(m) \mid \text{card } X \in \{r_0, \dots, r_{k-1}\}\}. \end{aligned}$$

endowed with containment.

Diagrams indexed by $B_m(\leq 2)$, for $m > 2$, are widely used in Gillibert [10].

As $B_m(\leq m) = B_m(\leq m - 1)$ for each positive integer m , we shall assume that $r < m$ whenever we consider the poset $B_m(\leq r)$. The following result is related to [8, Corollaire 3.3.3].

Proposition 5.2. *Let r and m be integers with $1 \leq r < m$ and let κ and λ be infinite cardinals. Then the following statements are equivalent:*

- (i) $(\kappa, < \lambda) \rightsquigarrow B_m(\leq r)$;
- (ii) for every $F: [\kappa]^{\leq r} \rightarrow [\kappa]^{< \lambda}$ there exists $H \in [\kappa]^m$ such that $F(X) \cap H \subseteq X$ for each $X \in [H]^{\leq r}$;
- (iii) $(\kappa, r, \lambda) \rightarrow m$ (cf. Section 1).

Proof. An easy argument, similar to the one used in the proof of Proposition 3.4, yields the equivalence of (i) and (ii). Furthermore, as every map from $[\kappa]^r$ to $[\kappa]^{< \lambda}$ trivially extends to a map from $[\kappa]^{\leq r}$ to $[\kappa]^{< \lambda}$, (ii) implies (iii).

Finally assume that (iii) holds, and let $F: [\kappa]^{\leq r} \rightarrow [\kappa]^{< \lambda}$. We must find an m -element free set, with respect to F , of κ . We may assume that F is isotone. By

applying (ii) to the restriction F' of F to $[\kappa]^r$, we obtain an m -element subset H of κ which is free with respect to F' . Let $X \in [H]^{\leq r}$. For each $\xi \in H \setminus X$, from $X \subseteq H \setminus \{\xi\}$ and $\text{card } X \leq r \leq \text{card}(H \setminus \{\xi\})$ it follows that there exists $Y \in [H]^r$ such that $X \subseteq Y \subseteq H \setminus \{\xi\}$. By applying this to all elements $\xi \in H \setminus X$, we obtain that we can write $X = \bigcap (X_i \mid i < s)$, for a positive integer s and subsets X_0, \dots, X_{s-1} of H . As $F(X_i) \cap H \subseteq X_i$ for each $i < s$ and as F is isotone, we obtain that $F(X) \cap H \subseteq X$. Therefore, H is free with respect to F , and so (ii) holds. \square

As $B_m(\leq r)$ is a finite lattice with breadth $r + 1$, it follows from Proposition 4.8 that $\text{kur } B_m(\leq r) \geq r + 1$, for all integers r and $m > r$ such that $1 \leq r < m$. Furthermore, as $B_m(\leq r)$ has exactly m join-irreducible elements (namely the singletons), it follows from Proposition 4.2 that $\text{kur } B_m(\leq r) \leq m$.

On the other hand, according to the results of Lázár, Hajnal-Máté, and Hajnal cited in Section 1, it follows from Proposition 5.2 that the relation $(\lambda^{+r}, < \lambda) \rightsquigarrow B_m(\leq r)$ holds for all integers $r \in \{1, 2, 3\}$ and $m > r$. In particular, $\text{kur } B_m(\leq r) \leq r + 1$, and thus, as the converse inequality holds, $\text{kur } B_m(\leq r) = r + 1$, whenever $r \in \{1, 2, 3\}$ and $r < m < \omega$. Arguing as above, in the presence of GCH we obtain from the relation $(\lambda^{+r}, r, \lambda) \rightarrow \lambda^+$ (cf. [6, Theorem 45.5]) that $\text{kur } B_m(\leq r) = r + 1$ —now for all integers r and m such that $1 \leq r < m$.

Without assuming GCH, Komjáth and Shelah's result (cf. Section 1) yields that $\text{kur } B_{t_4}(\leq 4)$ may be larger than or equal to 6. In particular, $\text{kur } B_{t_4}(\leq 4) = 5$ in any set-theoretical universe satisfying GCH, while $\text{kur } B_{t_4}(\leq 4) \geq 6$ in some generic extension. And therefore, the *Kuratowski index function is not absolute* (in the set-theoretical sense).

We sum up in the following proposition some of the results above.

Proposition 5.3.

- (i) $r + 1 \leq \text{kur } B_m(\leq r) \leq m$ for all integers r and m such that $1 \leq r < m$.
- (ii) $\text{kur } B_m(\leq r) = r + 1$ for all integers r and m such that $1 \leq r < m$ and $r \in \{1, 2, 3\}$.
- (iii) Assume that GCH holds. Then $\text{kur } B_m(\leq r) = r + 1$ for all integers r and m such that $1 \leq r < m$.
- (iv) There exists a model of ZFC where $\text{kur } B_{t_4}(\leq 4) \geq 6$.

While the integer t_4 of Proposition 5.3 is quite large, smaller values yield new Kuratowski indexes that were not available by using earlier methods. The following result is due to the first author [8, Lemme 3.3.12]. The proof of the order-dimension exact estimate $\text{dim } B_{n+2}(\leq n) = n + 1$ is contained in [5], however [8, Lemme 3.3.12] provides an easy direct proof in that case. Of course it is obvious that the breadth of $B_{n+2}(\leq n)$ is $n + 1$, and so the conclusion of Lemma 5.4 follows from Propositions 4.7 and 4.8.

Lemma 5.4. $\text{kur } B_{n+2}(\leq n) = \text{dim } B_{n+2}(\leq n) = \text{br } B_{n+2}(\leq n) = n + 1$, for every positive integer n .

The following corollary is observed in [8, Théorème 3.3.13].

Corollary 5.5 (Gillibert). *The relation $(\lambda^{+n}, n, \lambda) \rightarrow n + 2$ holds for each infinite cardinal λ and each positive integer n .*

In particular, this answers in the affirmative the question, raised on [6, page 285], whether $(\aleph_4, 4, \aleph_0) \rightarrow 6$.

These methods show that in order to prove the existence of large free sets for set mappings of type a positive integer r (as defined in [6, Section 46]), it is sufficient to establish upper bounds for the order-dimension of finite lattices of the form $B_m(\leq r)$. This problem gets somewhat simplified by using the following easy result.

Lemma 5.6. *The equality $\dim B_m(\leq r) = \dim B_m(1, r)$ holds, for all integers m and r such that $1 \leq r < m$.*

Proof. As $B_m(\leq r)$ contains $B_m(1, r)$, the inequality $\dim B_m(\leq r) \geq \dim B_m(1, r)$ is trivial. Conversely, set $N := \dim B_m(1, r)$ and let K be a product of chains (any lattice would do) with an order-embedding $\varphi: B_m(1, r) \hookrightarrow K$. We set

$$\psi(X) := \bigvee (\varphi(\{i\}) \mid i \in X), \quad \text{for each } X \in B_m(\leq r).$$

Clearly ψ is isotone. Let $X, Y \in B_m(\leq r)$ such that $\psi(X) \leq \psi(Y)$, we must prove that $X \subseteq Y$. We may assume that $Y \neq m$. In particular, $\text{card } Y \leq r$, and thus, by part of the proof of Proposition 5.2, we can write $Y = \bigcap (Y_j \mid j < s)$, for a positive integer s and $Y_0, \dots, Y_{s-1} \in [m]^r$. For each $i \in X$ and each $j < s$,

$$\varphi(\{i\}) \leq \psi(X) \leq \psi(Y) \leq \psi(Y_j) \leq \varphi(Y_j),$$

thus, as φ is an order-embedding, $i \in Y_j$. As this holds for all possible choices of i and j , we obtain that $X \subseteq Y$. Therefore, ψ is an order-embedding, and so $B_m(\leq r)$ order-embeds into the same product of chains as $B_m(1, r)$ does. \square

Getting estimates of the order-dimension of $B_m(1, r)$ has given rise to a great deal of work, starting with Dushnik [5]. Further refinements can be found, for example, in Kierstead [13, 14]. We shall illustrate how large free sets can be obtained from small order-dimensions in Proposition 5.7.

We shall use Dushnik's work [5]. For integers m, r with $1 \leq r \leq m$, Dushnik denotes by $N(m, r)$ the minimal number N such that there exists a set \mathcal{S} of N linear orderings of m such that for each $A \in [m]^r$ and each $a \in A$, there exists $S \in \mathcal{S}$ such that $(x, a) \in S$ for each $x \in A$. Then Dushnik establishes in [5, Theorem III] that

$$\dim B_m(1, r) = N(m, r + 1), \quad \text{for all integers } m, r \text{ such that } 1 < r < m. \quad (5.1)$$

In order to get estimates of $N(m, r + 1)$, we shall use Hajnal's work quoted in Spencer's paper [20]. For integers n and r such that $1 \leq r \leq n$, Spencer denotes by $M(n, r)$ the maximal cardinality of an " r -scrambling" family of subsets of n , and he establishes on [20, Lemma, p. 351] the inequality

$$M(n, r) \geq \left\lfloor \frac{1}{2} (1 - 2^{-r})^{-n/r} \right\rfloor, \quad (5.2)$$

where $\lfloor x \rfloor$ denotes the largest integer below any real number x . Furthermore, Spencer establishes on [20, page 351] the inequality

$$N(2^{M(n,r)}, r + 1) \leq n, \quad \text{for all integers } n, r \text{ such that } 1 \leq r < n. \quad (5.3)$$

Now let n and r be integers such that $1 < r < n$, and set

$$m := 2^{\lfloor \frac{1}{2} (1 - 2^{-r})^{-n/r} \rfloor}. \quad (5.4)$$

It follows from (5.2) that $2^{M(n,r)} \geq m$, thus, by the isotonicity of the function N , it follows from (5.3) that $N(m, r + 1) \leq N(2^{M(n,r)}, r + 1) \leq n$, and thus, by (5.1),

n	172	180	186	192	197	202	207	211	215
$E(n, 4)$	256	512	1,024	2,048	4,096	8,192	16,384	32,768	65,536

 TABLE 1. Small values of $E(n, 4)$

$\dim B_m(1, r) \leq n$. Therefore, by Lemma 5.6, $\dim B_m(\leq r) \leq n$ as well, and therefore, by Proposition 4.7, $\text{kur } B_m(\leq r) \leq n$. Now an immediate application of Proposition 5.2 yields the relation $(\kappa^{+(n-1)}, r, \kappa) \rightarrow m$, for each infinite cardinal κ . This completes the proof of the following result.

Proposition 5.7. *Let n and r be positive integers with $2 \leq r < n$. Then the relation $(\lambda^{+(n-1)}, r, \lambda) \rightarrow 2^{\lfloor \frac{1}{2}(1-2^{-r})^{-n/r} \rfloor}$ holds for each infinite cardinal λ .*

Denote by $\lg x$ and $\log x$ the base two logarithm, resp. the natural logarithm of a positive real number x . Set $E(n, r) := 2^{\lfloor \frac{1}{2}(1-2^{-r})^{-n/r} \rfloor}$, for all integers n, r such that $2 \leq r < n$. Elementary calculations yield the asymptotic estimate

$$\lg \lg E(n, r) \sim \frac{n}{r2^r \log 2}, \quad \text{as } n \gg r \gg 0.$$

We illustrate on Table 1 the behavior of the function E on $r := 4$ and (relatively) small values of n . The given values of n are minimal for the corresponding values of $E(n, 4)$: for example, $E(171, 4) = 128$ while $E(172, 4) = 256$. The value $n := 172$ is the first one for which Proposition 5.7 gives a nontrivial large free set result.

In particular, we deduce from Proposition 5.7 the following relations:

$$(\aleph_{210}, 4, \aleph_0) \rightarrow 32,768;$$

$$(\aleph_{214}, 4, \aleph_0) \rightarrow 65,536.$$

and so on. Smaller values of n also give nontrivial large free sets, however, for those values the bounds that we shall discuss now are better.

We recall the following estimate of the order-dimension of $B_m(1, r)$. This estimate is first stated explicitly in Brightwell *et al.* [2, Proposition 1.5], where it is attributed to Füredi and Kahn [7]. However, the estimate is not stated explicitly in the latter paper, although the proof is an easy modification of the proof of that paper's [7, Proposition 2.3].

Proposition 5.8 (Füredi and Kahn). *Let r and m be integers such that $1 \leq r < m$ and let d be a positive integer satisfying the inequality*

$$m \binom{m-1}{r} \left(\frac{r}{r+1} \right)^d < 1.$$

Then $\dim B_m(1, r) \leq d$.

If $\dim B_m(1, r) \leq d$, then, by Lemma 5.6, $\dim B_m(\leq r) \leq d$, thus, by Proposition 4.7, $\text{kur } B_m(\leq r) \leq d$, and thus, by Proposition 5.2, $(\aleph_{d-1}, r, \aleph_0) \rightarrow m$. In particular, we obtain the relation

$$(\aleph_{109}, 4, \aleph_0) \rightarrow 257,$$

which is better than the relation $(\aleph_{171}, 4, \aleph_0) \rightarrow 256$ given by Table 1. However, for $r = 4$ and $n \geq 211$ the size of the free set given by Proposition 5.7 becomes larger again.

We can even obtain further large free sets results by using Dushnik's original exact estimate of the order-dimension of a truncated cube under certain conditions [5].

Proposition 5.9 (Dushnik). *Let m , j , and k be positive integers such that $m \geq 4$, $2 \leq j \leq \lfloor \sqrt{m} \rfloor$, and*

$$\left\lfloor \frac{m + j^2 - j}{j} \right\rfloor \leq k < \left\lfloor \frac{m + (j - 1)^2 - j + 1}{j - 1} \right\rfloor.$$

Then $\dim B_m(1, k - 1) = m - j + 1$.

In the context of Proposition 5.9, it follows from Lemma 5.6 that $\dim B_m(\leq k - 1) = m - j + 1$, thus, by Proposition 4.7, $\text{kur} B_m(\leq k - 1) \leq m - j + 1$, and thus, by Proposition 5.2, the relation $(\aleph_{m-j}, k - 1, \aleph_0) \rightarrow m$ holds.

Interesting values are obtained for $k = 5$ and $m \in \{10, 11\}$, giving respectively the relations

$$(\aleph_7, 4, \aleph_0) \rightarrow 10 \quad \text{and} \quad (\aleph_8, 4, \aleph_0) \rightarrow 11.$$

For $k = 6$ and $m \in \{12, 13, 14\}$, we obtain, respectively,

$$(\aleph_9, 5, \aleph_0) \rightarrow 12, \quad (\aleph_{10}, 5, \aleph_0) \rightarrow 13, \quad (\aleph_{11}, 5, \aleph_0) \rightarrow 14.$$

Of course, the ‘‘origin’’ \aleph_0 can be changed, in any of these relations, to any infinite cardinal, so, for example, the relation $(\lambda^{+9}, 5, \lambda) \rightarrow 12$ holds for any infinite cardinal λ .

6. DISCUSSION

Our work [11] makes a heavy use of Kuratowski indexes of finite lattices, in particular in order to evaluate ‘‘critical points’’ between quasivarieties of algebraic structures. Now evaluating the Kuratowski index of a finite poset may be a hard problem, even in the case of easily describable finite lattices. For example, consider the finite lattices P and Q represented in Figure 6.1. Both P and Q have breadth

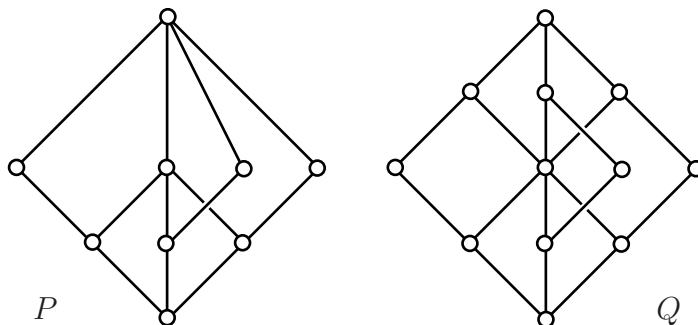


FIGURE 6.1. Two lattices of breadth two and order-dimension three

two and order-dimension three. Furthermore, P embeds, as a poset, into Q , thus, by Proposition 4.4, $\text{kur}(P) \leq \text{kur}(Q)$. Therefore, by Propositions 4.7 and 4.8, we obtain the inequalities

$$2 \leq \text{kur}(P) \leq \text{kur}(Q) \leq 3.$$

We do not know which one of those inequalities can be strengthened to an equality. The statement $\text{kur}(P) = 2$ is equivalent to

For every infinite cardinal λ and every $F: [\lambda^+]^{<\omega} \rightarrow [\lambda^+]^{<\lambda}$, there are distinct $\xi_0, \xi_1, \xi_2, \eta_0, \eta_1, \eta_2 < \lambda$ such that $\xi_i \notin F(\{\xi_j, \eta_j\})$, $\eta_i \notin F(\{\xi_j, \eta_j\})$, and $\eta_i \notin F(\{\xi_0, \xi_1, \xi_2\})$ for all $i \neq j$ in $\{0, 1, 2\}$,

while the statement $\text{kur}(Q) = 2$ is equivalent to the apparently stronger statement

For every infinite cardinal λ and every $F: [\lambda^+]^{<\omega} \rightarrow [\lambda^+]^{<\lambda}$, there are distinct $\xi_0, \xi_1, \xi_2, \eta_0, \eta_1, \eta_2 < \lambda$ such that $\xi_i \notin F(\{\xi_j, \eta_j\})$ and $\eta_i \notin F(\{\xi_0, \xi_1, \xi_2, \eta_j\})$ for all $i \neq j$ in $\{0, 1, 2\}$.

Such results could be relevant in further improving “large free sets” results such as Corollary 5.5 and Proposition 5.7. In particular, the gap between $E(n, 4)$ (which gives a positive large free set result in Proposition 5.7) and t_n (which gives Komjáth and Shelah’s negative large free set result) looks huge. Can the bound $E(n, 4)$ be improved in Proposition 5.7? On the other hand, we do not even know whether the relation $(\aleph_4, 4, \aleph_0) \rightarrow 7$ holds.

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